

# Identification Robust Inference for Risk Prices in Structural Stochastic Volatility Models

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## Abstract

In structural stochastic volatility asset pricing models, changes in volatility affect risk premia through two channels: (1) the investor’s willingness to bear high volatility in order to get high expected returns as measured by the market return risk price, and (2) the investor’s direct aversion to changes in future volatility as measured by the volatility risk price. Disentangling these channels is difficult and poses a subtle identification problem that invalidates standard inference. We adopt the discrete-time exponentially affine model of Han, Khrapov, and Renault (2018), which links the identification of the volatility risk price to the leverage effect. In particular, we develop a minimum distance criterion that links the market return risk price, the volatility risk price, and the leverage effect to well-behaved reduced-form parameters that govern the return and volatility’s joint distribution. The link functions are almost flat if the leverage effect is close to zero, making estimating the volatility risk price difficult. We translate the conditional quasi-likelihood ratio test that Andrews and Mikusheva (2016) develop in a nonlinear GMM framework to a minimum distance framework. The resulting conditional quasi-likelihood ratio test is uniformly valid. We invert this test to derive robust confidence sets that provide correct coverage for the risk prices regardless of the leverage effect’s magnitude.

**JEL Codes:** C12, C14, C38, C58, G12

**Keywords:** weak identification, robust inference, stochastic volatility, leverage, market return, risk premium, volatility risk premium, risk price, confidence set, asymptotic size

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# 1 Introduction

A fundamental question in finance is how investors optimally trade off risk and return. Economic theory predicts investors demand a higher return as compensation for bearing more risk. Hence, we should expect a positive relationship between the mean and volatility of returns. Some seminal early papers proposed a static trade-off between risk and expected return, most notably the capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965). In practice, volatility varies over time. Consequently, a significant strand of the recent literature examines the dynamic tradeoff between volatility and returns, including structural stochastic volatility models such as Christoffersen, Heston, and Jacobs (2013), Bansal et al. (2014), and Dew-Becker et al. (2017). In nonlinear models like these, investors care not just about how an asset's returns co-move with the volatility but also care how they co-move with changes in volatility.

In these structural stochastic volatility models, changes in volatility affect risk premia through two channels: (1) the investor's willingness to tolerate high volatility in order to get high expected returns as measured by the market return risk price, and (2) the investor's direct aversion to changes in future volatility as measured by the volatility risk price. We adopt the discrete-time exponentially affine model of Han, Khrapov, and Renault (2018), who represent the market return risk price and the volatility risk price by two structural parameters. In this model, Han, Khrapov, and Renault (2018) establish the significant result that the identification of the volatility risk price depends on a substantial leverage effect, which is the negative contemporaneous correlation between returns and volatility.

Although the leverage effect is theoretically less than zero, it is difficult to quantify empirically, and its estimate usually is small (Aït-Sahalia, Fan, and Li 2013). When the leverage effect is small, the data only provide a limited amount of information about the volatility risk, compared to the finite-sample noise in the data. This low signal-to-noise ratio, as modeled by weak identification, invalidates standard inference based on the generalized method of moments (GMM) estimator; see Stock and Wright (2000) and Andrews and Cheng (2012).

We provide an identification-robust confidence set for the structural parameters that measure the market return risk price, the volatility risk price, and the leverage effect. The robust confidence set provides correct asymptotic coverage, uniformly over a large set of models and allows for any magnitude of the leverage effect. This uniform validity is crucial for the confidence set to have good finite-sample coverage (Mikusheva 2007; Andrews and Guggenberger 2010). In contrast, standard confidence sets based on the GMM estimator and its asymptotic normality do not have uniform validity in the presence of a small leverage effect. This issue affects all of the structural parameters because they are estimated simultaneously.

We achieve robust inference in two steps. First, we establish a minimum distance criterion using link functions between the structural parameters and a set of reduced-form parameters that determine the joint distribution of the return and volatility. The structural model implies that the link functions are zero when evaluated at the true values of the structural parameters and the reduced-form parameters. Identification and estimation of these reduced form parameters

are standard and are not affected by the presence of a small leverage effect. However, the link functions are almost flat in one of the structural parameters when the leverage effect is small, resulting in weak identification. Second, given this minimum distance criterion, we invert the conditional quasi-likelihood ratio (QLR) test by Andrews and Mikusheva (2016) to construct a robust confidence set. The key feature of this test is that it treats the flat link functions as an infinite-dimensional nuisance parameter. The critical value is constructed by conditioning on a sufficient statistic for this nuisance parameter, and it is known to yield a valid test regardless of the nuisance parameter's value. Andrews and Mikusheva (2016) develop this test in a GMM framework. We show it works in minimum distance contexts such as the one considered here and provide conditions for its asymptotic validity. For practitioners, we provide a detailed algorithm for the construction of this simulation-based robust confidence set.

Our empirical results relates to the empirical analysis of the effect of volatility on risk premia. As Lettau and Ludvigson (2010) mention, the evidence here is inconclusive. Bollerslev, Engle, and Wooldridge (1988), Harvey (1989), Ghysels, Santa-Clara, and Valkanov (2005), Bali and Peng (2006), and Ludvigson and Ng (2007) find a positive relationship, while Campbell (1987), Breen, Glosten, and Jagannathan (1989), Pagan and Hong (1991), Whitelaw (1994), and Brandt and Kang (2004) find a negative relationship. Also, some papers use both a market return risk factor and a variance risk factor to explain the risk premia dynamics, including Christoffersen, Heston, and Jacobs (2013), Feunou et al. (2014), and Dew-Becker et al. (2017). In related strand of the literature, Bollerslev, Law, and Tauchen (2008) and Drechsler and Yaron (2011) document a substantial positive variance risk premium. We contribute to this literature by providing the first method for making valid inference for the market return risk price and the volatility risk price. This new confidence set not only allows for both effects but also takes into account the potential identification issue.

To have a non-linear relationship between changes in volatility and expected returns, we need either volatility of volatility (as used by Drechsler and Yaron (2011)) or jumps (as used by Drechsler (2013)). Since we are working in discrete-time, it is far more natural to work volatility than with jumps. This is because discontinuities are not well-defined in discrete-time; all functions are continuous in the discrete-topology. The most straightforward models that allow for closed-form expressions for the risk prices are exponentially-affine (not affine) models. This is why they are frequently used in the option-pricing literature. In order to avoid complicating the analysis, we use such a model. We focus on the time-series behavior of the index because as a complement to, not a subtitle to, estimating the risk prices from cross-sectional or options pricing data. Using market-level variation over time to examine this non-linear relationship is a common approach in the literature, used by both the variance-premium literature and Han, Khrapov, and Renault (2018).

The weak identification issue studied in this paper is relevant in many economic applications, ranging from linear instrumental variable models (Staiger and Stock 1997) to nonlinear structural models (Mavroeidis, Plagborg-Møller, and Stock 2014; Andrews and Mikusheva 2015). This paper

is the first one to study this issue in structural asset pricing models with stochastic volatility.

Moreira (2003) introduces the conditional inference approach to the linear instrumental variable model creating the conditional likelihood-ratio (CLR) test, and Kleibergen (2005) applies it to the nonlinear GMM problem. Magnusson and Mavroeidis (2010) and Magnusson (2010) extend Kleibergen’s (2005) results to the minimum-distance case. The key issue with these papers is that they rely exclusively upon local behavior of the moment-conditions. This is inherently under-powered in some environments. Andrews and Mikusheva (2016) resolve this issue by developing a global approach in the GMM case by proposing conditional inference for nonlinear GMM problems with an infinite-dimensional nuisance parameter. Their method is known to be the most-powerful in some special cases. We develop a global weak-identification robust inference method for minimum distance estimation by extending Andrews and Mikusheva (2016). We bear the same relationship to Magnusson and Mavroeidis (2010) and Magnusson (2010) that Andrews and Mikusheva (2016) bears to Kleibergen (2005). We also extend the scope of the application of these weak-identification robust methods to a new type of asset pricing model with substantial non-linearity and heteroskedasticity.

The rest of the paper is organized as follows. Section 2 provides the model and its parameterization. Section 3 provides model-implied restrictions and use them to derive the link function. Section 4 provides the asymptotic distribution of the reduced-form parameter and robust confidence sets for the structural parameter. A detailed algorithm to construct the robust confidence set is given in Section 4.3. Section 5 show that the method works well in simulation, and Section 6 applies the methods to data on the S&P 500 providing estimates of the risk prices. Section 7 concludes. Proofs are given in the appendix.

## 2 Model

This section provides a parametric structural model with stochastic volatility, following Han, Khrapov, and Renault (2018). They extend the discrete-time exponentially-affine model of Darolles, Gouriéroux, and Jasiak (2006), and their model is a natural discrete-time analog of the Heston (1993) model. We specify this model using a stochastic discount factor (SDF), also called the pricing kernel, and the physical measure, which gives the joint distribution of the return and volatility dynamics.<sup>1</sup> We first define the SDF and parameterize it as an exponential affine function with unknown parameters. We then provide parametric distribution for the physical measure.

Let  $P_t$  be the price of the asset under consideration. Let  $r_{t+1} = \log(P_{t+1}/P_t) - r_f$  denote the log excess return minus the risk-free rate and  $\sigma_{t+1}^2$  denote the associated volatility. The observed data is  $W_t = (r_t, \sigma_t^2)$  for  $t = 1, \dots, T$ . Let  $\mathcal{F}_t$  be the representative investor’s information set at time  $t$ .

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<sup>1</sup>The risk-neutral measure is unobserved due to the lack of option data.

## 2.1 Stochastic Discount Factor and Its Parameterization

The prices of all assets satisfy the following asset pricing equation in terms of the SDF:

$$P_t = \mathbb{E}[M_{t,t+1} \exp(-r_f) P_{t+1} | \mathcal{F}_t]. \quad (1)$$

Following the definition of  $r_{t+1}$ , the pricing equation implies that for all assets

$$1 = \mathbb{E}[M_{t,t+1} \exp(r_{t+1}) | \mathcal{F}_t]. \quad (2)$$

We start by parameterizing the SDF by the exponential affine model. Let  $\pi$  be the price of volatility risk and  $\kappa$  be the market return risk price. They are both considered as structural parameters.

**Definition 1.** Parameterizing the Stochastic Discount Factor

$$M_{t,t+1}(\pi, \kappa) = \exp(m_0 + m_1 \sigma_t^2 - \pi \sigma_{t+1}^2 - \kappa r_{t+1}). \quad (3)$$

Throughout we assume that the two risks that command nonzero prices are the market return risk price and the volatility risk price. These two risks are closely related to the first two moments of  $r_{t+1}$ . Consequently, we only use variation in the first two moments of the data to estimate these parameters.

## 2.2 Parameterizing the Volatility and Return Dynamics

Next, we parameterize the joint distribution of  $\{W_t : t = 1, \dots, T\}$ . Following Han, Khrapov, and Renault (2018), we make the following assumptions. First, the return  $r_t$  and volatility  $\sigma_t^2$  are first-order Markov. Second, there is no Granger-causality from the return to the volatility. Third, returns are independent across time given the volatility. We do allow  $\sigma_t^2$  and  $r_t$  to be contemporaneously correlated, as they are in the data.

Under these assumptions, the volatility drives all of the dynamics of the process. The only relevant information in the information set  $\mathcal{F}_t$  for time  $t + 1$ -measurable variables is contained in  $\sigma_t^2$ . In general,  $\sigma_t^2$ ,  $\sigma_{t+1}^2$ , and  $r_{t+1}$  form a sufficient statistic for  $\mathcal{F}_{t+1}$ .

We adopt the conditional autoregressive gamma process as in Gouriéroux and Jasiak (2006) and Han, Khrapov, and Renault (2018) for the volatility process. The model is parameterized in terms of the Laplace transform:

$$\mathbb{E}[\exp(-x\sigma_{t+1}^2) | \mathcal{F}_t] = \exp(-A(x)\sigma_t^2 - B(x)) \quad (4)$$

for all  $x \in \mathbb{R}$ . The function  $A(x)$  and  $B(x)$  are parameterized as follows.

**Definition 2.** Parameterize the Volatility Dynamics

$$A(x) := \frac{\rho x}{1 + cx}, \quad (5)$$

$$B(x) := \delta \log(1 + cx), \quad (6)$$

with  $\rho \in [0, 1 - \epsilon]$ ,  $c > \epsilon$ ,  $\delta > \epsilon$  for some  $\epsilon > 0$ .

In this specification,  $\rho$  is a persistence parameter,  $c$  is a scale parameter, and  $\delta$  is a level parameter. We can see this clearly in the following conditional mean and variance formulas for  $\sigma_{t+1}^2$ .

**Remark 1** (Volatility Moment Conditions).

$$\mathbb{E} [\sigma_{t+1}^2 \mid \sigma_t^2] = \rho \sigma_t^2 + c\delta, \quad (7)$$

$$\mathbb{V}\text{ar} [\sigma_{t+1}^2 \mid \sigma_t^2] = 2c\rho\sigma_t^2 + c^2\delta. \quad (8)$$

Next, we model the return dynamics. Similar to the volatility dynamics, the distribution of  $r_t$  given both  $\sigma_{t+1}^2$  and  $\sigma_t^2$  is specified in terms of the Laplace transform:

$$\mathbb{E} [\exp(-xr_{t+1}) \mid \mathcal{F}_t, \sigma_{t+1}^2] = \exp(-C(x)\sigma_{t+1}^2 - D(x)\sigma_t^2 - E(x)) \quad (9)$$

for all  $x \in \mathbb{R}$ . The function  $C(x)$ ,  $D(x)$ , and  $E(x)$  are parameterized as follows such that the return has a conditional Gaussian distribution.

**Definition 3.** Parameterizing the Return Dynamics

$$C(x) := \psi x - \frac{1 - \phi^2}{2} x^2, \quad (10)$$

$$D(x) := \beta x, \quad (11)$$

$$E(x) := \gamma x, \quad (12)$$

with  $\phi \in [-1 + \epsilon, 0]$  for some  $\epsilon > 0$ .

Under this specification, we have the following representation of the conditional mean and variance for  $r_{t+1}$ .

**Remark 2** (Return Moment Conditions).

$$\mathbb{E} [r_{t+1} \mid \sigma_t^2, \sigma_{t+1}^2] = \psi \sigma_{t+1}^2 + \beta \sigma_t^2 + \gamma, \quad (13)$$

$$\mathbb{V}\text{ar} [r_{t+1} \mid \sigma_t^2, \sigma_{t+1}^2] = (1 - \phi^2) \sigma_{t+1}^2. \quad (14)$$

The parameter  $\phi$  represents the leverage effect because it measures the reduction in return's volatility caused by conditioning on the volatility path.

### 3 Link Functions

So far, we have introduced the following parameters:  $(m_0, m_1, \kappa, \pi)$  in SDF,  $(\rho, c, \delta)$  for the volatility dynamics, and  $(\psi, \beta, \gamma, \phi)$  for the return dynamics. Next, we explore restrictions among these parameters that are consistent with this model. In other words, not all of these parameters can change freely under the structural model.

We use these restrictions to construct link functions between a set of reduced-form parameters and a set of structural parameters. These link functions play an important role in separating the regularly behaved reduced-form parameters from the structural parameters. They also are used to conduct identification robust inference for the structural parameters based on a minimum distance criterion. All of these restrictions are also imposed in the GMM estimation in Han, Khrapov, and Renault (2018). However, because the volatility risk price is weakly identified, they calibrate it instead of estimating it. Given this calibrated value, they proceed to estimate all other parameters with GMM.

#### 3.1 Pricing Equation Restrictions

We first explore restrictions implied by the pricing equation  $\mathbb{E}[M_{t,t+1} \exp(r_{t+1}) | \mathcal{F}_t] = 1$ . We start with a simple result stating that the constants  $m_0$  and  $m_1$  are normalization constants implied by all the other parameters. Thus,  $m_0$  and  $m_1$  are not free parameters to be estimated. Instead, they should take the values given below, once other parameters are specified. These restrictions on  $m_0$  and  $m_1$  are obtained by applying the restriction  $\mathbb{E}[M_{t,t+1} \exp(r_{t+1}) | \mathcal{F}_t] = 1$  to the risk free asset. Applying the same argument to any other asset, we also obtain another set of two restrictions, which can be written in terms of the coefficients  $\beta$  and  $\gamma$  under the linear form of  $D(x)$  and  $E(x)$ .

**Lemma 1.** *Given the parameterization in the model, the pricing equation*

$\mathbb{E}[M_{t,t+1} \exp(r_{t+1}) | \mathcal{F}_t] = 1$  *implies that*<sup>2</sup>

$$\begin{aligned} m_0 &= E(\kappa) + B(\pi + C(\kappa)), \\ m_1 &= D(\kappa) + A(\pi + C(\kappa)), \end{aligned}$$

and

$$\begin{aligned} \gamma &= B(\pi + C(\kappa - 1)) - B(\pi + C(\kappa)), \\ \beta &= A(\pi + C(\kappa - 1)) - A(\pi + C(\kappa)). \end{aligned}$$

The two equalities on  $\beta$  and  $\gamma$  link them to the market return risk price,  $\kappa$ , and the volatility risk price,  $\pi$ , through the functions  $A(\cdot), B(\cdot), C(\cdot)$ , which also involve the parameters  $(\rho, c, \delta, \psi, \phi)$ . We treat these two equalities as link functions in the minimum distance criterion specified below.

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<sup>2</sup>Proof A.1

### 3.2 Leverage Effect Restrictions

Following Han, Khrapov, and Renault (2018), we parameterize  $\psi$  as

$$\psi = \frac{\phi}{\sqrt{2c}} - \frac{1 - \phi^2}{2} + (1 - \phi^2)\kappa. \quad (15)$$

The first part  $\phi/\sqrt{2c}$  measures the leverage effect arising from the instantaneous correlation between  $r_{t+1}$  and  $\sigma_{t+1}^2$ . The second part is the traditional Jensen effect term that arises from taking expectation of a log-Gaussian random variable. The third term arises from risk-aversion, which is why it is proportional to  $\kappa$ .

### 3.3 Structural and Reduced-Form Parameters

Because  $\phi$  is the leverage effect parameter, we group it together with market return risk price,  $\kappa$ , and the volatility risk price,  $\pi$ , and call  $\theta := (\kappa, \pi, \phi)'$  structural parameters. These structural parameters are estimated by restrictions from this structural model. In contrast, the other parameters in the conditional mean and variance of the return and volatility, see Remark 1 and Remark 2, are simply estimated using these moments, without any model restrictions. As such, we call them the reduced-form parameters. Because  $1 - \phi^2$  shows up in the conditional variance of  $r_{t+1}$ , see (14), we define  $\zeta = 1 - \phi^2$  as a reduced-form parameter and link it to the structural parameter  $\phi$  through this relationship. To sum up, the reduced-form parameters are  $\omega := (\rho, c, \delta, \psi, \beta, \gamma, \zeta)'$ .

Using  $\zeta$  as a reduced-form parameter has the additional benefit of avoiding estimating  $\phi$  directly. Estimating  $\phi$  when its true value is close to 0 results in an estimator with a non-standard asymptotic distribution due to the boundary constraint. The inference procedure below does not require estimation of  $\phi$  and is uniform over  $\phi$  even if its true value is on or close to the boundary 0. It is worth noting that this boundary condition gives us additional information in estimating  $\phi$  in some cases. The estimator for  $\phi$  may converge quite rapidly; however, it is almost certainly not asymptotically approximately Gaussian. In addition, we cannot recover asymptotic Gaussianity by removing this constraint. Even though,  $\phi$  could conceivably be greater than 0,  $\phi^2$  cannot conceivably be less than 0. The  $\phi$  parameter enters the last link in (16) through  $\phi^2$ . This is where the non-standard behavior arises. Economically, we saying that the variance of  $r_{t+1}$  must reduce when we condition on more information. Although, this is clearly in population, it may not hold for the sample variances.

The link functions between the structural parameter  $\theta$  and the reduced-form parameter  $\omega$  are collected together in

$$g(\theta, \omega) = \begin{pmatrix} \gamma - [B(\pi + C(\kappa - 1)) - B(\pi + C(\kappa))] \\ \beta - [A(\pi + C(\kappa - 1)) - A(\pi + C(\kappa))] \\ \psi - (1 - \phi^2)\kappa + \frac{1}{2}(1 - \phi^2) - 1/(2c)^{1/2}\phi \\ \zeta - (1 - \phi^2) \end{pmatrix}. \quad (16)$$



For the inference problem studied below, we know  $g(\theta_0, \omega_0) = 0$  when evaluated at the true value of  $\theta$  and  $\omega$ .

### 3.4 Identification

One of the important contributions of Han, Khrapov, and Renault (2018) is to establish the relationship between the identification of the volatility risk price and the leverage effect. In particular, they show that when the leverage effect parameter  $\phi = 0$ , the volatility risk price  $\pi$  is not identified. To see this result, note that the only source of identification information on  $\pi$  are the first two link functions in  $g(\theta_0, \omega_0) = 0$ , which come from Lemma 1. Clearly, these two equations are independent of  $\pi$  if  $C(\kappa) = C(\kappa - 1)$ . Using the definition of  $C(\cdot)$  and (15), we have

$$C(\kappa) - C(\kappa - 1) = \psi - (1 - \phi^2) \left( \kappa - \frac{1}{2} \right) = \frac{\phi}{\sqrt{2c}}. \quad (17)$$

Clearly, the strength of identification is governed by the strength of the leverage effect. In other words, we need  $\phi \neq 0$  to identify the volatility risk price  $\pi$ .

Even if  $\phi \neq 0$ , we do not know it. In practice, with a finite-sample size and different types of noise in the data, such as measurement errors and omitted variables, a substantial leverage effect is required to obtain a standard identification situation where the noise in the data is negligible compared to the information to identify  $\pi$ . However, if only a small leverage effect is found, as in Bandi and Renò (2012) and Aït-Sahalia, Fan, and Li (2013), or the magnitude of the leverage effect is completely unknown, an identification robust procedure is needed to conduct inference in this problem. In addition, standard minimum-distance estimators do not provide valid inference when some of the first-stage parameters are either asymptotically non-Gaussian or the link functions are ill-behaved. In our case, we should not expect  $\phi$  to be asymptotically Gaussian even though it is well identified. We provide a procedure that is robust to both non-standard issues now.

## 4 Robust Inference for Risk Prices

### 4.1 Asymptotic Distribution of the Reduced-Form Parameter

Write  $\omega = (\omega_1, \omega_2, \omega_3)'$ , where  $\omega_1 = (\rho, c, \delta)' \in O_1$ ,  $\omega_2 = (\gamma, \beta, \psi)' \in O_2$ , and  $\omega_3 = \zeta \in O_3$ . The parameter space for  $\omega$  is  $O = O_1 \times O_2 \times O_3 \subset R^{d_\omega}$ . The true value of  $\omega$  is assumed to be in the interior of the parameter space.

Below we describe the estimator  $\hat{\omega} := (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)'$  and provide its asymptotic distribution. We estimate these parameters separately because  $\omega_1$  only shows up in the conditional mean and variance of  $\sigma_{t+1}^2$ ,  $\omega_2$  only shows up in the conditional mean of  $r_{t+1}$ , and  $\omega_3$  only shows up in the conditional variance of  $r_{t+1}$ .

We first estimate  $\omega_1 = (\rho, c)'$  based on the conditional mean and variance of  $\sigma_{t+1}^2$ , which can

be equivalently written as

$$\begin{aligned} E[\sigma_{t+1}^2|\sigma_t^2] &= A \text{ and } E[\sigma_{t+1}^4|\sigma_t^2] = B, \text{ where} \\ A &= \rho\sigma_t^2 + c\delta \text{ and } B = A^2 + (2c\rho\sigma_t^2 + c^2\delta). \end{aligned} \quad (18)$$

Because the conditional mean of  $\sigma_{t+1}^2$  and  $\sigma_{t+1}^4$  are linear and quadratic functions, respectively, of the conditioning variable  $\sigma_t^2$ , they can be transformed to the unconditional moments

$$E[h_t(\omega_{10})] = 0, \text{ where } h_t(\omega_1) = [(1, \sigma_t^2) \otimes (\sigma_{t+1}^2 - A), (1, \sigma_t^2, \sigma_t^4) \otimes (\sigma_{t+1}^4 - B)]', \quad (19)$$

and  $\omega_{10}$  represents the true value of  $\omega_1$ . The two-step GMM estimator of  $\omega_1$  is

$$\hat{\omega}_1 = \arg \min_{\omega_1 \in O_1} \left( T^{-1} \sum_{t=1}^T h_t(\omega_1) \right)' \hat{V}_1 \left( T^{-1} \sum_{t=1}^T h_t(\omega_1) \right), \quad (20)$$

where  $\hat{V}_1$  is a consistent estimator of  $V_1 := \sum_{m=-\infty}^{\infty} \text{Cov}[h_t(\omega_{10}), h_{t+m}(\omega_{10})]$ .

We estimate  $\omega_2$  by the generalized least squares (GLS) estimator because the conditional mean of  $r_{t+1}$  is a linear function of the conditioning variable  $\sigma_t^2$  and  $\sigma_{t+1}^2$  and the conditional variance is proportional to  $\sigma_{t+1}^2$ . The GLS estimator of  $\omega_2$  is

$$\begin{aligned} \hat{\omega}_2 &= \left( \sum_{t=1}^T x_t x_t' \right)^{-1} \sum_{t=1}^T x_t y_t, \text{ where} \\ x_t &= \sigma_{t+1}^{-1} (1, \sigma_t^2, \sigma_{t+1}^2)' \text{ and } y_t = \sigma_{t+1}^{-1} r_{t+1}. \end{aligned} \quad (21)$$

We estimate  $\omega_3$  by the sample variance estimator:

$$\hat{\omega}_3 = T^{-1} \sum_{t=1}^T (y_t - \hat{y}_t)^2, \text{ where } \hat{y}_t = x_t' \hat{\omega}_2. \quad (22)$$

Let  $P$  denote the distribution of the data  $\{W_t = (r_{t+1}, \sigma_{t+1}^2) : t \geq 1\}$  and  $\mathcal{P}$  denote the parameter space of  $P$ . Note that the true values of the structural parameter and the reduced-form parameters are all determined by  $P$ . We allow  $P$  to change with  $T$ . For notational simplicity, the dependence on  $P$  and  $T$  is suppressed.

Let

$$f_t(\omega) = \begin{pmatrix} h_t(\omega_1) \\ x_t(y_t - x_t'\omega_2) \\ (y_t - x_t'\omega_2)^2 \end{pmatrix} \in R^{df} \text{ and } V = \sum_{m=-\infty}^{\infty} \text{Cov}[f_t(\omega_0), f_{t+m}(\omega_0)]. \quad (23)$$

The estimator  $\hat{\omega}$  defined above is based on the first moment of  $f_t(\omega)$ . Thus, the limiting distribution of  $\hat{\omega}$  relates to the limiting distribution of  $T^{-1/2} \sum_{t=1}^T (f_t(\omega_0) - \mathbb{E}[f_t(\omega_0)])$  following from the central limit theorem. Furthermore, because  $\omega_1$  is the GMM estimator based on some nonlinear moment conditions, we need uniform convergence of the sample moments and their derivatives to show the

consistency and asymptotic normality of  $\widehat{\omega}_1$ . These uniform convergence follows from the uniform law of large numbers. Because  $\widehat{\omega}_2$  is a simple OLS estimator by regressing  $y_t$  and  $x_t$ , we need the regressors to not exhibit multicollinearity. We make the necessary assumptions below. All of them are easily verifiable with weakly dependent time series data.

Let  $\widehat{V}$  denote a heteroskedasticity and autocorrelation consistent (HAC) estimator of  $V$ . The estimator  $\widehat{V}_1$  is a submatrix of  $\widehat{V}$  associate with  $V_1$ . Let  $H_t(\omega_1) = \partial h_t(\omega_1)/\partial \omega_1'$ .

**Assumption R.** The following conditions hold uniformly over  $P \in \mathcal{P}$ , for some fixed  $0 < C < \infty$ .

1.  $T^{-1} \sum_{t=1}^T (h_t(\omega_1) - \mathbb{E}[h_t(\omega_1)]) \rightarrow_p 0$  and  $T^{-1} \sum_{t=1}^T (H_t(\omega_1) - \mathbb{E}[H_t(\omega_1)]) \rightarrow_p 0$ ,  $\mathbb{E}[H_t(\omega_1)]$  is continuous in  $\omega_1$ , all uniformly over the parameter space of  $\omega_1$ .
2.  $T^{-1} \sum_{t=1}^T (x_t x_t' - \mathbb{E}[x_t x_t']) \rightarrow_p 0$ .
3.  $V^{-1/2} \{T^{-1/2} (\sum_{t=1}^T f_t(\omega_0) - \mathbb{E}[f_t(\omega_0)])\} \rightarrow_d N(0, I)$  and  $\widehat{V} - V \rightarrow_p 0$ .
4.  $C^{-1} \leq \lambda_{\min}(A) \leq \lambda_{\max}(A) \leq C$  for  $A = V, \mathbb{E}[H_t(\omega_{1,0})' H_t(\omega_{1,0})], \mathbb{E}[x_t x_t'], \mathbb{E}[z_t z_t']$ , where  $z_t = (1, \sigma_t^2, \sigma_t^4)'$ .<sup>3</sup>

Let  $H(\omega_1) = \mathbb{E}[H_t(\omega_1)]$  and  $\overline{H}(\omega_1) = T^{-1} \sum_{t=1}^T H_t(\omega_1)$ . Define

$$\begin{aligned} \mathcal{B} &= \text{diag}\{[H(\omega_{10})V_1^{-1}H(\omega_{10})]^{-1}H(\omega_{10})V_1^{-1}, \mathbb{E}[x_t x_t']^{-1}, 1\}, \\ \widehat{\mathcal{B}} &= \text{diag}\{[\overline{H}(\widehat{\omega}_1)' \widehat{V}_1^{-1} \overline{H}(\widehat{\omega}_1)]^{-1} \overline{H}(\widehat{\omega}_1)' \widehat{V}_1^{-1}, [T^{-1} \sum_{t=1}^T x_t x_t']^{-1}, 1\}. \end{aligned} \quad (24)$$

The following lemma provides the asymptotic distribution of the reduced-form parameter and a consistent estimator of its asymptotic covariance. Note, we put the asymptotic covariance on the left side of the convergence to allow the distribution of the data to change with sample size  $T$ .

**Lemma 2.** *Suppose Assumption R holds. The following results hold uniformly over  $P \in \mathcal{P}$ .<sup>4</sup>*

$$\xi_T := \Omega^{-1/2} T^{-1/2} (\widehat{\omega} - \omega_0) \rightarrow_d \xi \sim N(0, I), \text{ where } \Omega = \mathcal{B}V\mathcal{B}',$$

and

$$\widehat{\Omega} - \Omega \rightarrow_p 0, \text{ where } \widehat{\Omega} = \widehat{\mathcal{B}}\widehat{V}\widehat{\mathcal{B}}'.$$

## 4.2 Weak Identification

The true value of the structural parameter  $\theta$  and the reduced-form parameter  $\omega$  satisfy the link function  $g(\theta_0, \omega_0) = 0$ . In a standard problem without any identification issues, we can estimate  $\theta_0$  by the minimum distance estimator  $\widehat{\theta} = (\widehat{\kappa}, \widehat{\pi}, \widehat{\phi})'$ , which minimizes  $Q_T(\theta) = g(\theta, \widehat{\omega})' W_T g(\theta, \widehat{\omega})$  for some weighting matrix  $W_T$ , and construct tests and confidence sets for  $\theta_0$  using an asymptotically

<sup>3</sup>We use  $\lambda(\text{matrix})$  to denote the eigenvalue of the matrix.

<sup>4</sup>Proof A.2

normal approximation for  $T^{1/2}(\widehat{\theta} - \theta_0)$ . However, this standard method does not work in the present problem when  $\pi_0$  is only weakly identified. In this case,  $g(\theta, \widehat{\omega})$  is almost flat in  $\pi$  and the minimum distance estimator of  $\widehat{\pi}$  is not even consistent. To make the problem even more complicated, the inconsistency of  $\widehat{\pi}$  has a spillover effect on  $\widehat{\kappa}$  and  $\widehat{\phi}$ , making the distribution of  $\widehat{\kappa}$  and  $\widehat{\phi}$  non-normal even in large samples.

Before presenting the robust confidence set, we first introduce some useful quantities and provide a heuristic discussion of the identification problem and its consequences. Let  $G(\theta, \omega)$  denote the partial derivative of  $g(\theta, \omega)$  with respect to (w.r.t.)  $\omega$ . Let  $g_0(\theta) = g(\theta, \omega_0)$  and  $G_0(\theta) = G(\theta, \omega_0)$  be the link function and its derivative evaluated at  $\omega_0$  and  $\widehat{g}(\theta) = g(\theta, \widehat{\omega})$  and  $\widehat{G}(\theta) = G(\theta, \widehat{\omega})$  be the same quantities evaluated at the estimator  $\widehat{\omega}$ . The delta method gives

$$\eta_T(\theta) := T^{1/2} [\widehat{g}(\theta) - g_0(\theta)] = G_0(\theta)\Omega^{1/2} \cdot \xi_T + o_p(1), \quad (25)$$

where  $\xi_T \rightarrow_d N(0, I)$  following [Lemma 2](#). Thus,  $\eta_T(\cdot)$  weakly converges to a Gaussian process  $\eta(\cdot)$  with covariance function  $\Sigma(\theta_1, \theta_2) = G_0(\theta_1)\Omega G_0(\theta_2)'$ .

Following [\(25\)](#), we can write  $T^{1/2}\widehat{g}(\theta) = \eta_T(\theta) + T^{1/2}g_0(\theta)$ , where  $\eta_T(\theta)$  is the noise from the reduced-form parameter estimation and  $T^{1/2}g_0(\theta)$  is the signal from the link function. Under weak identification,  $g_0(\theta)$  is almost flat in  $\theta$ , modeled as the signal  $T^{1/2}g_0(\theta)$  being finite even for  $\theta \neq \theta_0$  and  $T \rightarrow \infty$ . Thus, the signal and the noise are of the same order of magnitude, yielding an inconsistent minimum distance estimator  $\widehat{\theta}$ . This is in contrast with the strong identification scenario, where  $T^{1/2}g_0(\theta) \rightarrow \infty$  for  $\theta \neq \theta_0$  as  $T \rightarrow \infty$  and  $g_0(\theta_0) = 0$ . In this case, the signal is strong enough that the minimum distance estimator is consistent.

The identification strength of  $\theta_0$  is determined by the function  $T^{1/2}g_0(\theta)$ . However, this function is unknown and cannot be consistently estimated (due to  $T^{1/2}$ ). Thus, we take the conditional inference procedure as in [Andrews and Mikusheva \(2016\)](#) and view  $T^{1/2}g_0(\theta)$  as an infinite dimensional nuisance parameter for the inference of  $\theta_0$ . The goal is to construct robust confidence set for  $\theta_0$  that has correct size asymptotically regardless of this unknown nuisance parameter.

### 4.3 Conditional QLR Test

We construct a confidence set for  $\theta \in \Theta := [0, M_1] \times [-M_2, 0] \times [1 - \epsilon, 0]$  by inverting the test  $H_0 : \theta = \theta_0$  vs  $H_1 : \theta \neq \theta_0$ , where  $M_1$  and  $M_2$  are large positive constants and  $\epsilon$  is a small positive constant. The test statistic is a QLR statistic that takes the form

$$QLR(\theta_0) := T\widehat{g}(\theta_0)' \widehat{\Sigma}(\theta_0, \theta_0)^{-1} \widehat{g}(\theta_0) - \min_{\theta \in \Theta} T\widehat{g}(\theta)' \widehat{\Sigma}(\theta, \theta)^{-1} \widehat{g}(\theta), \quad (26)$$

where  $\widehat{\Sigma}(\theta_1, \theta_2) = \widehat{G}(\theta_1)\widehat{\Omega}\widehat{G}(\theta_2)'$  and  $\widehat{\Omega}$  is the consistent estimator of  $\Omega$  defined above.

[Andrews and Mikusheva \(2016\)](#) provide the conditional QLR test in a nonlinear GMM problem, where  $\widehat{g}(\theta)$  is replaced by a sample moment. The same method can be applied to the present nonlinear minimum distance problem. Following [Andrews and Mikusheva \(2016\)](#), we first project

$\widehat{g}(\theta)$  onto  $\widehat{g}(\theta_0)$  and construct a residual process

$$\widehat{r}(\theta) = \widehat{g}(\theta) - \widehat{\Sigma}(\theta, \theta_0) \widehat{\Sigma}(\theta_0, \theta_0)^{-1} \widehat{g}(\theta_0). \quad (27)$$

The limiting distributions of  $\widehat{r}(\theta)$  and  $\widehat{g}(\theta_0)$  are Gaussian and independent. Thus, conditional on  $\widehat{r}(\theta)$ , the asymptotic distribution of  $\widehat{g}(\theta)$  no longer depends on the nuisance parameter,  $T^{1/2}g_0(\theta)$ , making the procedure robust to any identification strength.

Specifically, we obtain the  $1-\alpha$  conditional quantile of the QLR statistic, denoted by  $c_{1-\alpha}(r, \theta_0)$ , as follows. For  $b = 1, \dots, B$ , we take independent draws  $\eta_b^* \sim N(0, \widehat{\Sigma}(\theta_0, \theta_0))$  and produce a simulated process,

$$g_b^*(\theta) := \widehat{r}(\theta) + \widehat{\Sigma}(\theta, \theta_0) \widehat{\Sigma}(\theta_0, \theta_0)^{-1} \eta_b^*, \quad (28)$$

and a simulated statistic,

$$QLR_b^*(\theta_0) := T \eta_b^{*'} \widehat{\Sigma}(\theta_0, \theta_0)^{-1} \eta_b^* - \min_{\theta \in \Pi} T g_b^*(\theta)' \widehat{\Sigma}(\theta, \theta)^{-1} g_b^*(\theta). \quad (29)$$

Let  $b_0 = \lceil (1-\alpha)B \rceil$ , the smallest integer greater than or equal to  $(1-\alpha)B$ . Then the critical value  $c_{1-\alpha}(r, \theta_0)$  is the  $b_0^{th}$  smallest value among  $\{QLR_b^*, b = 1, \dots, B\}$ . We execute the steps reported in [Algorithm 1](#) to form a robust confidence set for  $\theta$ .

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**Algorithm 1** Construing the Confidence Set

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1. Estimate the reduced-form parameter  $\widehat{\omega} = (\widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3)'$  following the estimators defined in [\(20\)](#), [\(21\)](#), and [\(22\)](#).
2. Obtain a consistent estimator of  $\widehat{\omega}$ 's asymptotic covariance  $\widehat{\Omega} = \widehat{B} \widehat{V} \widehat{B}'$ , where  $\widehat{B}$  is defined in [\(24\)](#) and  $\widehat{V}$  is a HAC estimator of  $V$ .
3. For  $\theta_0 \in \Theta$ ,
  - (a) Construct the QLR statistic  $QLR(\theta_0)$  in [\(26\)](#) using  $g(\theta, \omega)$ ,  $G(\theta, \omega)$ ,  $\widehat{\omega}$ , and  $\widehat{\Omega}$ .
  - (b) Compute the residual process  $\widehat{r}(\theta)$  in [\(27\)](#).
  - (c) Given  $\widehat{r}(\theta)$ , compute the critical value  $c_{1-\alpha}(r, \theta_0)$  described above.
4. Repeat these steps for different values of  $\theta_0$ . Construct a confidence set by collecting the null values that are not rejected, i.e., the nominal level  $1-\alpha$  confidence set for  $\theta_0$  is

$$CS_T = \{\theta_0 : QLR_T(\theta_0) \leq c_{1-\alpha}(r, \theta_0)\}.$$


---

To obtain confidence intervals for each element of  $\theta_0$ , one simple solution is to project the confidence set constructed above to each axis. The resulting confidence interval also has correct coverage. An alternative solution is to first concentrate out the nuisance parameters before applying the conditional inference approach above, see Andrews and Mikusheva ([2016](#), Section 5). However, this concentration approach only works when the nuisance parameter is strongly identified. In the

present set-up, this approach does not work for  $\kappa$  and  $\phi$  because the nuisance parameter  $\pi$  is weakly identified.

**Assumption S.** The following conditions hold over  $P \in \mathcal{P}$ , for any  $\theta$  in its parameter space, and any  $\omega$  in some fixed neighborhood around its true value, for some fixed  $0 < C < \infty$ .

1.  $g(\theta, \omega)$  is partially differentiable in  $\omega$ , with partial derivative  $G(\theta, \omega)$  that satisfies  $\|G(\theta_1, \omega) - G(\theta_2, \omega)\| \leq C\|\theta_1 - \theta_2\|$  and  $\|G(\theta, \omega_1) - G(\theta, \omega_2)\| \leq C\|\omega_1 - \omega_2\|$ .
2.  $C^{-1} \leq \lambda_{\min}(G(\theta, \omega)'G(\theta, \omega)) \leq \lambda_{\max}(G(\theta, \omega)'G(\theta, \omega)) \leq C$ .

**Theorem 3.** Suppose *Assumption R* and *Assumption S* hold. Then,

$$\liminf_{T \rightarrow \infty} \inf_{P \in \mathcal{P}} \Pr(\theta_0 \in CS_T) \geq 1 - \alpha.^5$$

This theorem states that the confidence set constructed by the conditional QLR test has correct asymptotic size. Uniformity is important for this confidence set to cover the true parameter with a probability close to  $1 - \alpha$  in finite-samples. This uniform result is established over a parameter space  $\mathcal{P}$  that allows for weak identification of the structural parameter  $\theta$ .

## 5 Simulations

In this section, we investigate the finite-sample performance of the proposed test and show that the asymptotic approximations derived above work well in practice. We also compare it with the standard test that assumes all parameters are strongly identified. The standard test is known to be invalid under weak identification but its degree of distortion is unknown in general. We simulate the data with the parametric model above where the true values of the parameters are given in [Table 1](#) based on the values used by Han, Khrapov, and Renault (2018). To investigate the robustness of the procedure with respect to various identification strengths, we vary both  $\phi$  and  $T$ . Specifically, we consider  $\phi \in \{-0.40, -0.10, -0.01\}$  and  $T \in \{2,000; 10,000\}$ . The number of data points in the empirical section is 3,700.

Table 1: Simulation Set-up

$\delta$	$\rho$	$c$	$\pi$	$\kappa$
Parameter Values used by Han, Khrapov, and Renault (2018)				
0.6475	0.50	$3.94128 \times 10^{-3}$	-10	1.7680

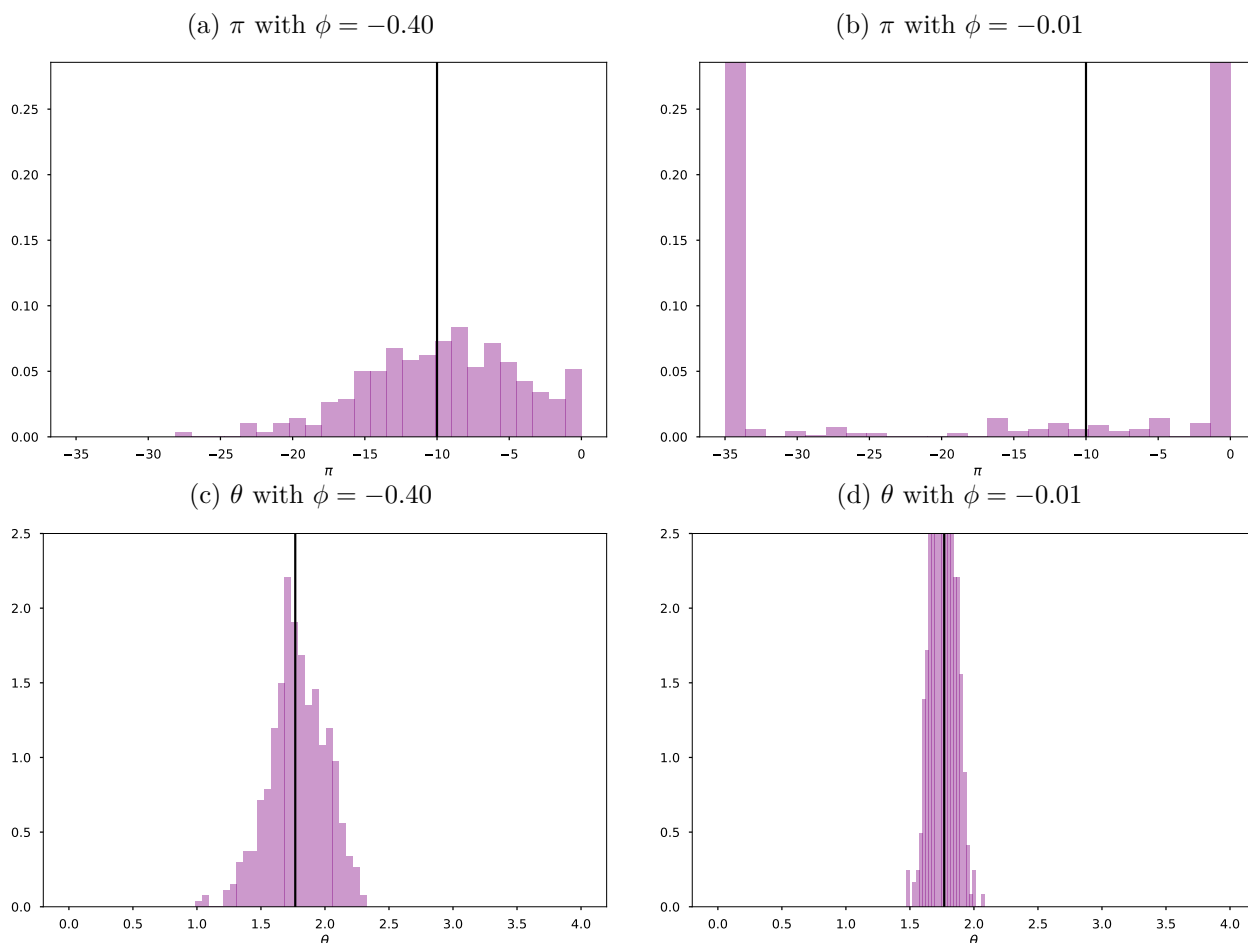
To avoid boundary issues with respect to the estimate of  $c$  and  $\delta$  in finite-sample, we reparameterize the moment conditions and link functions in terms of  $\log(c)$ ,  $\log(c) + \log(\delta)$ , and  $\text{logit}(\rho)$ . This reparameterization forces the scale parameters to be positive and  $\rho$  to lie in  $(0, 1)$ . We find

<sup>5</sup>Proof A.3

that the resulting estimates for the transformed reduced-form parameters are better approximated by the Gaussian distribution for a given finite sample.

To show the effect of various identification strength, we first vary the true value of  $\phi$  and plot the distribution of  $\hat{\pi}$  and  $\hat{\theta}$  in Figure 1. The reported result is based on 10,000 observations and 500 simulation repetitions. The black lines in the middle of the figures are the true parameter values. Clearly, the estimators sometimes pile up at the boundaries of the parameter space. As expected, this simulation shows that the Gaussian distribution is not a good approximation for the finite-sample distribution of either of the estimators.

Figure 1: Parameter Estimates'  $t$ -Statistics



Next, we study the finite-sample size of in the standard QLR test and the proposed conditional QLR test for a joint test for the three structural parameters. The nominal level of the test is 5%. The critical value of the standard QLR test is the 95% quantile of the  $\chi^2$ -distribution with 3 degree of freedom. The critical value of the conditional QLR test is obtained by the stimulation-based procedure in Algorithm 1, with 250 simulation repetitions to approximate the quantile of the conditional distribution. The finite-sample size is based on 250 simulation repetitions.

The standard test is no longer valid under weak identification because the QLR statistic does

not have a  $\chi^2$ -distribution in this case. However, it is not clear whether the standard QLR test under-rejects or over-rejects in finite-sample and how large is the difference from 5%.

Table 2: Finite-Sample Size of the Standard and Proposed Tests

	Standard %	Proposed %	Standard %	Proposed %
$\phi$	$T = 2,000$		$T = 10,000$	
-0.01	2.00	5.20	1.60	4.40
-0.40	2.40	5.60	6.00	6.40

Simulation results show that the standard QLR test under-rejects in finite-sample. This is most severe when the identification is weak, e.g., for  $\phi = -0.01$  and  $T = 10,000$ , the rejection rate is 1.6%. If we have enough data and  $\phi$  is large enough in magnitude, the standard test static does okay. However, this is not the empirically relevant case. The proposed test, however, has approximately uniform coverage and, hence, is much more trustworthy.

## 6 Data and Empirical Results

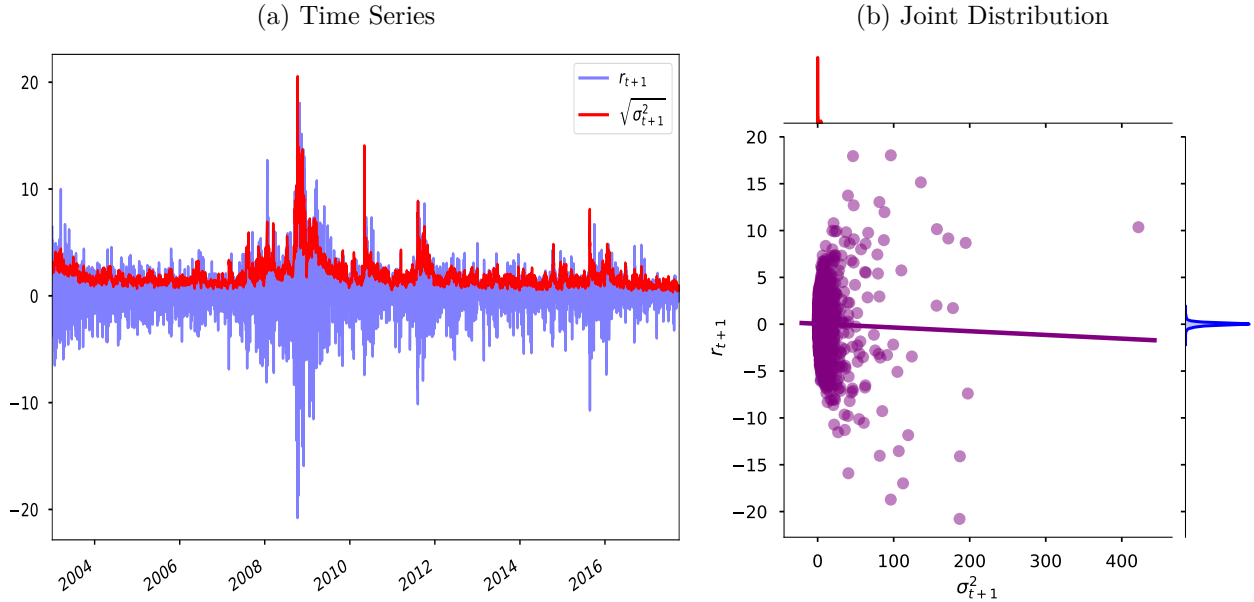
For the empirical application, we use the daily return on the S&P 500 for  $r_{t+1}$  and the associated realized volatility computed with high-frequency data for  $\sigma_{t+1}^2$ . The data is obtained from SPY (SPDR S&P 500 ETF Trust), an exchange-traded fund that mimics the S&P 500. This gives us a market index whose risk is not easily diversifiable and can be used to estimate the prices of risk that investors face in practice. We use the procedure Sangrey (2018) develops to estimate the integrated total volatility, i.e, the instantaneous expectation of the price variance. This measure reduces to the integrated diffusion volatility if prices have continuous paths and it works well in the presence of market microstructure noise.

Since SPY is one of the most liquid assets traded, we can choose the frequency at which we sample the underlying price. To balance market-microstructure noise, computational cost, and efficiency of the resultant estimators, we sample at the 1-second frequency. We annualize the data by multiplying  $r_{t+1}$  by 252 and  $\sigma_{t+1}^2$  by  $252^2$ . The data starts in 2003 and ends in September 2017. Since the asset is only traded during business hours, this leads to 3713 days of data with an average of approximately 24 000 observations per day. We compute  $r_{t+1}$  as the daily return from the open to the close of the market, the interval over which we can estimate the volatility. This avoids specifying the relationship between overnight and intra-day returns. We preprocess the data using the pre-averaging approach as in Podolskij and Vetter (2009) and Ait-Sahalia, Jacod, and Li (2012).

To see how the data move over time, we plot their time series in Figure 2. We also plot the joint unconditional distribution in Figure 2 to see the static relationship between the two series. The volatility has a long-right tail, a typical gamma-type distribution. The returns has a bell-shaped distribution. They are slightly negatively correlated, as shown by the regression line in the joint



Figure 2: S&P 500 Volatility and Log-Return



plot. This corroborates the work by Bandi and Renò (2012) and Ait-Sahalia, Fan, and Li (2013). We also report a series of summary statistics.

Table 3: Summary Statistics

	$r_{t+1}$	$\sigma_{t+1}^2$
Mean	0.02	5.62
Standard Deviation	2.35	14.46
Skewness	-0.31	12.21
Kurtosis	13.07	243.40
Correlation	-0.02	

We now report the estimates and confidence intervals for the reduced-form parameters  $c$ ,  $\delta$ , and  $\rho$ . The confidence intervals reported here use the Gaussian limiting theory, i.e., the point estimates  $\pm 1.96$  standard errors. We first obtain confidence intervals for  $\log(c)$  and  $\log(c) + \log(\delta)$ , and transform them into confidence intervals for  $c$  and  $\delta$ . Similarly, we create the confidence interval for  $\rho$  by inverting the interval for  $\text{logit}(\rho)$ .

Table 4: Parameters that Govern the Volatility Process

	Point Estimate	95 % Confidence Interval
$c$	3.07	(1.38, 6.79)
$\delta$	37.98	(17.65, 81.72)
$\rho$	0.77	(0.67, 0.85)

For confidence intervals of the three structural parameters, we first compute their joint confi-

dence set based on the conditional QLR test and then project it to each of the components. We also plot a joint confidence sets for the two risk prices, after projecting out  $\phi$ . We use 500 simulations to compute the quantile for the QLR statistic.

Table 5: Structural Parameters

	95 % Confidence Interval
$\phi$	(-0.33, -0.27)
$\pi$	(-30.97, 0.00)
$\kappa$	(0.00, 2.00)

The results in [Table 5](#). have a few notable features. First, we can reject the null hypothesis  $\phi = 0$ . We cannot, however, reject the hypothesis that  $\pi = 0$  at the 5 % level. We cannot reject the null hypotheses that  $\kappa = 0$ . This should not be particularly surprising given the difficulty in precisely estimating this parameter documented in the previous literature, ([Lettau and Ludvigson 2010](#)). Although not recorded on the table, we can reject the hypothesis that both  $\kappa = \pi = 0$ . The procedure can determine that investors demand compensation for risk, just not what combination of risks they demand compensation for. The risk price associated with the market return is covered by (0.00, 2.00) with at least 95 % probability. The volatility risk price is covered by (-30.97, 0.00) with at least 95 % probability. Our confidence intervals for both parameters are reasonable given other values that previous authors have found. For example, [Han, Khrapov, and Renault \(2018\)](#) preferred a value of  $\pi = -10$ , for example.

## 7 Conclusion

In structural stochastic volatility models as the one considered here, changes in the volatility affect returns through two channels. On the one hand, investors are willing to tolerate high volatility to get high expected returns as measured by the price of market return risk. On the other hand, investors are directly averse to changes in future volatility, as measured by the price of volatility risk. [Han, Khrapov, and Renault \(2018\)](#) shows how to disentangle these two channels by exploiting information arising from the leverage effect in an exponentially-affine pricing model. However, standard inference for this structural model is invalid because the volatility risk price is only weakly identified when the leverage effect is mild. This paper propose an identification robust inference procedure that provides reliable confidence sets for the risk prices regardless of the magnitude of the leverage effect. We take this procedure to the data on the S&P 500. The robust inference procedure provides reliable yet informative confidence intervals for the risk prices associated with the market return and the volatility.

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## Appendix A Proofs

### A.1 Proof of Lemma 1

*Proof.* For the risk-free asset, the excess return  $r_{t+1} = 0$ . Therefore, we have

$$\begin{aligned}
 1 &= E \left[ \exp \left( m_0 + m_1 \sigma_t^2 - \pi \sigma_{t+1}^2 - \theta r_{t+1} \right) \mid \mathcal{F}_t \right] \\
 &= \exp(m_0 + m_1 \sigma_t) E \left[ \exp \left( -\pi \sigma_{t+1}^2 \right) E \left[ \exp \left( -\theta r_{t+1} \right) \mid \mathcal{F}_t, \sigma_{t+1}^2 \right] \mid \mathcal{F}_t \right] \\
 &= \exp(m_0 - E(\theta) + m_1 \sigma_t - D(\theta) \sigma_t^2) E \left[ \exp \left( -\pi \sigma_{t+1}^2 - C(\theta) \sigma_{t+1}^2 \right) \mid \mathcal{F}_t \right] \\
 &= \exp(m_0 - E(\theta) + m_1 \sigma_t - D(\theta) \sigma_t^2 - A(\pi + C(\theta)) \sigma_t^2 - B(\pi + C(\theta))),
 \end{aligned}$$

where the first equality follows from the pricing equation, the second equality follows from the law of iterated expectations, the third equation uses the Laplace transform for  $r_{t+1}$  in (9), and the last equality follows from the Laplace transform for  $\sigma_{t+1}^2$  in (4). Since  $M_{t,t+1}$  must integrate to 1, the constant term and coefficient for  $\sigma_t^2$  must equal 0, which gives the claimed result for  $m_0$  and  $m_1$ .

We can apply the same argument above to any asset  $r_{t+1}$ . This gives the same result, except  $\theta$  is replaced by  $\theta - 1$  throughout. This implies that the two equalities for  $m_0$  and  $m_1$  also hold with  $\theta$  replaced by  $\theta - 1$ . Therefore,

$$\begin{aligned} E(\theta - 1) + B(C(\theta - 1) + \pi) &= E(\theta) + B(C(\theta) + \pi), \\ D(\theta - 1) + A(C(\theta - 1) + \pi) &= D(\theta) + A(C(\theta) + \pi). \end{aligned}$$

The claimed results for  $\gamma$  and  $\beta$  follow from  $\gamma = \mathbb{E}(\theta) - E(\theta - 1)$  and  $\beta = D(\theta) - D(\theta - 1)$  under the linear specification of  $E(x) = \gamma x$  and  $D(x) = \beta x$ .

□

## A.2 Proof of Lemma 2

*Proof.* Under the assumption that (i)  $\mathbb{E}(z_t z_t')$  has the smallest eigenvalue bounded away from 0 and (ii)  $c > \varepsilon$  and  $\delta > \varepsilon$  for some  $\varepsilon > 0$ , we not only have  $\omega_{10}$  as an unique minimizer of  $\|\mathbb{E}[h_t(\omega_1)]\|$  but also have a uniform positive lower bound for  $\|\mathbb{E}[h_t(\omega_1)]\|$  for  $\|\omega_1 - \omega_{10}\| \geq \varepsilon$ . Thus, consistency of  $\hat{\omega}_1$  follows from standard arguments for the consistency of a GMM estimator under an uniform convergence of the criterion under [Assumption R](#) (1) and (2).

Let  $\bar{h}(\omega_1) = T^{-1} \sum_{t=1}^T h_t(\omega_1)$  and  $\bar{H}(\omega) = T^{-1} \sum_{t=1}^T H_t(\omega)$ . By construction, the estimator satisfies the first order condition

$$\begin{aligned} 0 &= \begin{pmatrix} \bar{H}(\hat{\omega}_1)' \hat{V}_1^{-1} \bar{h}(\hat{\omega}_1) \\ T^{-1} \sum_{t=1}^T x_t (y_t - x_t' \hat{\omega}_2) \\ \hat{\omega}_3 - T^{-1} \sum_{t=1}^T (y_t - \hat{y}_t)^2 \end{pmatrix} \\ &= \begin{pmatrix} \bar{H}(\hat{\omega}_1)' \hat{V}_1^{-1} \bar{h}(\omega_{10}) + \bar{H}(\hat{\omega}_1)' \hat{V}_1^{-1} \bar{H}(\tilde{\omega}_1) (\hat{\omega}_1 - \omega_{10}) \\ T^{-1} \sum_{t=1}^T x_t (y_t - x_t' \omega_{20}) - T^{-1} \sum_{t=1}^T x_t x_t' (\hat{\omega}_2 - \omega_{20}) \\ (\hat{\omega}_3 - \omega_3) + \omega_3 - T^{-1} \sum_{t=1}^T (y_t - x_t \hat{\omega}_2)^2 \end{pmatrix}, \end{aligned} \quad (30)$$

where the second equality follows from a mean value expansion of  $\bar{h}(\hat{\omega}_1)$  around  $\omega_{10}$ , with  $\tilde{\omega}_1$  between  $\omega_{10}$  and  $\hat{\omega}_1$ . Let

$$\tilde{\mathcal{B}} = \text{diag} \left\{ [\bar{H}(\hat{\omega}_1)' \hat{V}_1^{-1} \bar{H}(\tilde{\omega}_1)]^{-1} \bar{H}(\hat{\omega}_1)' \hat{V}_1^{-1}, [T^{-1} \sum_{t=1}^T x_t x_t']^{-1}, 1 \right\}. \quad (31)$$

Then (30) implies that

$$\begin{aligned}
T^{1/2}(\widehat{\omega} - \omega) &= \widetilde{\mathcal{B}} \cdot T^{-1/2} \sum_{t=1}^T \begin{pmatrix} -h_t(\omega_{10}) \\ x_t(y_t - x'_t \omega_{20}) \\ (y_t - x'_t \widehat{\omega}_2)^2 - \omega_3 \end{pmatrix} \\
&= \widetilde{\mathcal{B}} \cdot T^{-1/2} \sum_{t=1}^T \begin{pmatrix} -h_t(\omega_{10}) \\ x_t(y_t - x'_t \omega_{20}) \\ (y_t - x'_t \omega_{20})^2 - \mathbb{E}[(y_t - x'_t \omega_{20})^2] \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \varepsilon_T \end{pmatrix}, \tag{32}
\end{aligned}$$

where the second equality uses  $\omega_3 = \mathbb{E}[(y_t - x'_t \omega_{20})^2]$  by definition and

$$\begin{aligned}
\varepsilon_T &= T^{-1/2} \sum_{t=1}^T \left[ (y_t - x'_t \widehat{\omega}_2)^2 - (y_t - x'_t \omega_{20})^2 \right] \\
&= 2T^{-1} \sum_{t=1}^T (y_t - x'_t \omega_{20}) x'_t \left[ T^{1/2} (\widehat{\omega}_2 - \omega_{20}) \right] + o_p(1) \\
&= o_p(1) \tag{33}
\end{aligned}$$

because  $T^{-1} \sum_{t=1}^T (y_t - x'_t \omega_{20}) x'_t \rightarrow_p 0$  and  $T^{1/2}(\widehat{\omega}_2 - \omega_{20}) = O_p(1)$  following [Assumption R](#). In addition,

$$\widetilde{\mathcal{B}} \rightarrow_p \mathcal{B} \tag{34}$$

following from the consistency of  $\widehat{\omega}_1$  and [Assumption R](#). Finally, the desirable result follows from (32)–(34) and [Assumption R](#). The consistency of  $\widehat{\Omega}$  follows from the consistency of  $\widehat{\mathcal{B}}$  and  $\widehat{V}$ .  $\square$

### A.3 Proof of [Theorem 3](#)

*Proof.* We obtain this result by applying Andrews and Mikusheva ([2016](#), Theorem 1). We now verify Assumptions 1–3 in Andrews and Mikusheva ([2016](#)). To show weak convergence  $\eta_T(\cdot)$  to  $\eta(\cdot)$  uniformly over  $\mathcal{P}$ , note that by a second-order Taylor expansion,

$$\begin{aligned}
\eta_T(\lambda) &:= T^{1/2} [\widehat{g}(\lambda) - g_0(\lambda)] = G_0(\lambda) \Omega^{1/2} \xi_T + \delta_T, \text{ where} \\
\xi_T &= \Omega^{-1/2} T^{1/2} (\widehat{\omega} - \omega_0), \text{ and } \delta_T = (G(\lambda, \widetilde{\omega}) - G(\lambda, \omega_0)) T^{1/2} (\widehat{\omega} - \omega_0) \tag{35}
\end{aligned}$$

and  $\widetilde{\omega}$  is between  $\widehat{\omega}$  and  $\omega_0$ . Because  $\|G(\lambda, \widetilde{\omega}) - G(\lambda, \omega_0)\| \leq C \|\widetilde{\omega} - \omega_0\|$ ,  $\delta_T = o_p(1)$  uniformly over  $\mathcal{P}$  following [Lemma 2](#). To show  $G_0(\lambda) \Omega^{1/2} \xi_T$  weakly converges to  $\eta(\cdot)$ , it is sufficient to show (i) the pointwise convergence

$$\begin{pmatrix} G_0(\lambda_1) \Omega^{1/2} \xi_T \\ G_0(\lambda_2) \Omega^{1/2} \xi_T \end{pmatrix} \rightarrow_d \begin{pmatrix} \eta(\lambda_1) \\ \eta(\lambda_2) \end{pmatrix}, \tag{36}$$

which follows from [Lemma 2](#), and (ii) the stochastic equicontinuity condition, i.e., for every  $\varepsilon > 0$  and  $\xi > 0$ , there exists a  $\delta > 0$  such that

$$\limsup_{T \rightarrow \infty} \Pr \left( \sup_{P \in \mathcal{P}} \sup_{\|\lambda_1 - \lambda_2\| \leq \delta} \left\| G_0(\lambda_1) \Omega^{1/2} \xi_T - G_0(\lambda_2) \Omega^{1/2} \xi_T \right\| > \varepsilon \right) < \xi. \quad (37)$$

For some  $C < \infty$ , we have  $\|G_0(\lambda_1) - G_0(\lambda_2)\| \leq C \|\lambda_1 - \lambda_2\|$  under a uniform bound for the derivative in [Assumption S](#), and we have  $\|\Omega^{1/2}\| \leq C$  under [Assumption R](#) because  $F$  and  $V$  both have bounded largest eigenvalue. Thus,

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \Pr \left( \sup_{P \in \mathcal{P}} \sup_{\|\lambda_1 - \lambda_2\| \leq \delta} \left\| G_0(\lambda_1) \Omega^{1/2} \xi_T - G_0(\lambda_2) \Omega^{1/2} \xi_T \right\| > \varepsilon \right) \\ & \leq \limsup_{T \rightarrow \infty} \Pr \left( C^2 \sup_{P \in \mathcal{P}} \|\xi_T\| > \frac{\varepsilon}{\delta} \right). \end{aligned} \quad (38)$$

Because  $\xi_T = O_p(1)$  uniformly over  $P \in \mathcal{P}$ , there exists  $\delta$  such that  $\varepsilon/\delta$  is large enough to make the right hand side of the inequality in [\(38\)](#) smaller than  $\xi$ .

Assumptions 2 and 3 of Andrews and Mikusheva ([2016](#), Theorem 1) follow from [Assumption R](#). □