Jumps, Realized Densities, and News Premia

Paul Sangrey

University of Pennsylvania
paul@sangrey.io

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Return Dynamics

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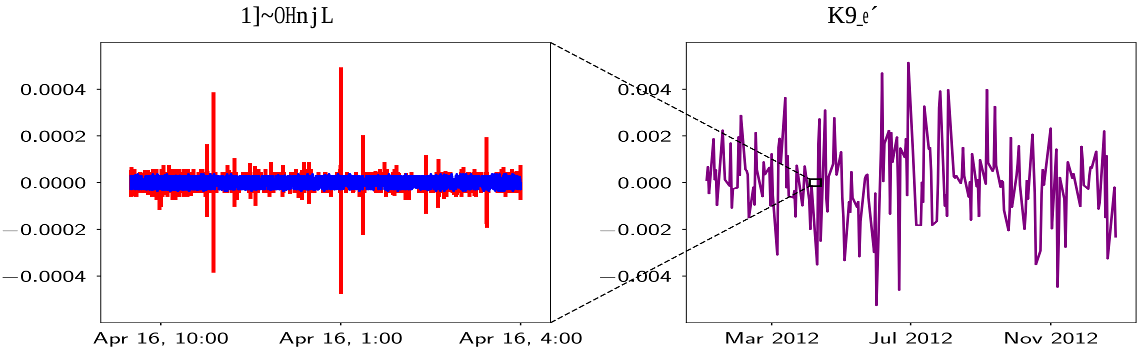

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What Characterizes Investors' Time-Varying Risk?

What summarizes the information returns' time-varying densities: $\backslash(\}_{\S}jF_{\S\quad 1})$**?**

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When prices are continuous, the diffusion volatility — $^{2}(\S)$ — does.

1. Volatility summarizes distributional dynamics.
2. We can estimate $^{2}(\S)$ using realized volatility or bipower variation.
3. Expected returns are instantaneous covariances.

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If prices jump, we need an entire density — A Realized Density.

1. Existing jump variation measures don't summarize distributional dynamics.
2. We can't estimate time-varying tail risk.
3. Expected returns aren't just instantaneous covariances.

Talk Structure / Contributions

- Propose a new measure — jump volatility

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Talk Structure / Contributions

[illegible] Propose a new measure — jump volatility

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Talk Structure / Contributions

Ou Propose a new measure — jump volatility

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Jumps := Discontinuities in the Price Process.

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- Price changes caused by discrete (possibly small) releases of information.
- They may come at unpredictable times.
- They may be observed by only a few investors.

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- FOMC Announcements, (Andersen, Bollerslev, Diebold, and Vega 2003) and (Engelberg 2008; Gürkaynak, Kısacıkoğlu, and Wright 2018, WP).
- A startup announcing a new product line.
- Effectively anything in a Bloomberg or Associated Press feed relevant for asset pricing.
- Private communications between investors.

Literature Review

1. **Modeling prices in continuous-time:**
   9s **The stochastic volatility diffusion case:**
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   @s **Allowing for jumps (We need infinitely many):**
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2. **Representing prices in continuous-time:**
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Literature Review

1. **Modeling prices in continuous-time:**
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     ?9}jLn}¸]i_0e 0j 9jL ~\0q\9}L r©··©sJ 8jL0} 0jJ ?nee0} e0®J K_0@neLJ 9jL d9@´ r©··¨sJ 9jL g´ce9jL 9jL µ\9jX r©··ks
   - @s **Allowing for jumps** (We need infinitely many)**:**
     ?9}jLn}¸]i_0e 0j 9jL ~\0q\9}L r©··VsJ 8{§]~9\9e_9 9jL `9HnL r©··k9J ©··k@J ©·o©sJ ?nee0} e0® 9jL ¦nLn}n® r©·oosJ 9jL W9ee9j§ 9jL ¦9«H\0j r©·oPs
2. **Representing prices in continuous-time:**
   - 9s K9h@_ rok£UsJ K«@_j 9jL ~H\°9}¶ rok£UsJ K«¹0J p9jJ 9jL ~_jXe0§nj r©···sJ 9jL ¦nLn}n® 9jL ¦9«H\0j r©·oVs
3. **Modeling and measuring volatility and news premia:**
   - 9s **Modeling the continuous case:**
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   - @s **Measuring volatility and news premia:**
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   - Hs **Allowing for jumps (in information):**
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Data Generating Process

p}_H0 p}nH0 rK_¸« _njs

Wiener Process

/ T(i) = / q(i)

Diffusion (Lq^{K})

Data Generating Process

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Diffusion Volatility   Wiener Process

/ T(i) = (i) / q(i)

Diffusion (Lq^{K})

Data Generating Process

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Diffusion Volatility | Wiener Process | Poisson Process

$$/ T(i) = (i) / q(i) + / L(i)$$

Diffusion (Lq^{K}) | Jumps ($Lq^{`}$)

Data Generating Process

p}_HO p}nHO r~§nH\9 §_H -ne9§_e_§´ `«hq K_¸« _njs

Diffusion Volatility — Wiener Process — Jump Magnitudes — Poisson Random Measure

$$/\,T(\,i) = \underbrace{(\,i)\ /\,q(\,i)}_{\text{Diffusion } (Lq^{K})} + \underbrace{\int_{R} (\,i;\ t)(\,M\quad)(\,/\,i;\ /\,t)}_{\text{Jumps } (Lq^{`})}$$

Data Generating Process

p}_HO p}nHO r~§nH\9 §_H -ne9§_e_§´ `«hq K_¸« _njs

Diffusion Volatility — Wiener Process — Jump Magnitudes — Poisson Random Measure

$$/ T(i) = \underbrace{(i) / q(i)}_{\text{Diffusion } (Lq^K)} + \underbrace{\int_R (i; t)(M)(/ i; / t)}_{\text{Jumps } (Lq^{`})}$$

|0§«}j

$$`_§ := \int_{§ 1}^{§} / T( b); \quad `_§ j F_{§ 1} \quad ?(`_§ j F_{§ 1})$$

The Literature Focuses on Models of the Form:

$$\grave{}_{\S} j F_{\S \ 1} \qquad ?(\grave{}_{\S} j F_{\S \ 1}) = \int_{{}^2{}_{\S}} 7(\grave{}_{\S} j \ t_{\S}) \ / : (t_{\S} j F_{\S \ 1})$$

Sufficient statistic (factor) for the dynamics

How should we model $\backslash(\}_{\S} j F_{\S \ 1})$? !

- What should we use for ${}^2{}_{\S}$?
- What should we use for T?
- What should we use for W?

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$\}_{\S} \qquad {}_{\S}\, i\,(0;1)$

$H\,Q^2_{\S}; = \quad H\,Q^2_{\S}; {}_1 + \quad i\,(0;1)$

- ${}^2{}_{\S}$ is the volatility ${}^2_{\S}$.
- $T \ \}_{\S} \ {}^2_{\S}$ is $i\,(0; {}^2_{\S})$.
- W is $8\,|\,(1)$.

Previous Work (Diffusion)

- ne9§_e_§´

Merton (1973)

$$^2(\S) := \lim_{! \, 0} \frac{1}{} E \left[\int_{\S}^{\S +} \left(Lq^{K}(\ ) \right)^2 F_{\S} \right]$$

Previous Work (Diffusion)

-ne9§_e_§´

Merton (1973)

$$ {}^{2}(\S) := \underset{!\,0}{\mathrm{H\,B}}\ \frac{1}{}\ \mathrm{KE}\left[\int_{\S}^{\S+} Lq^{K}(\ )^{2}\ F_{\S}\right] $$

|09e_¶0L K0j _§´

Barndorff-Nielsen and Shephard (2002),
Andersen, Bollerslev, Diebold, and Labys (2003)

$$ |K_{\S} := T\ \}_{\S}\ \ {}^{2}_{\S} = T\ {}_{{}^{2}_{\S}=[\int_{\S\ 1}^{\S} {}^{2}(\ )L]} = i\ \ 0;\int_{\S\ 1}^{\S} {}^{2}(\ )L $$

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Diffusion Volatility

$^2(§) :=$ H B $\frac{1}{}$KE R §+ § $Lq^{K}(\ )^{2}$ $F_{§}$! 0

Details

This Paper (Jump Diffusion)

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Diffusion Volatility

$$^{2}(\S) := \mathrm{H}\ \mathrm{B}\frac{1}{}\mathrm{KE}_{!\ 0} \ \mathrm{R}^{\S+}_{\S} \ Lq^{K}(\ )^{2} \ F_{\S}$$

Jump Volatility

$$^{2}(\S) := \mathrm{H}\ \mathrm{B}\frac{1}{}\mathrm{KE}_{!\ 0} \ \mathrm{R}^{\S+}_{\S} \ Lq^{`}(\ )^{2} \ F_{\S}$$

Details

This Paper (Jump Diffusion)

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Diffusion Volatility

$^{2}(§) := \underset{!\ 0}{H}\ B\frac{1}{}KE \quad R_{§}^{§+} \quad Lq^{K}(\ )^{2} \quad F_{§}$

Jump Volatility

$^{2}(§) := \underset{!\ 0}{H}\ B\frac{1}{}KE \quad R_{§}^{§+} \quad Lq^{`}(\ )^{2} \quad F_{§}$

|09e_¶0L K0j _§´I

(Derived Below)

$| K_{§} := T\ \}_{§} \quad {}^{2}_{§};\ {}^{2}_{§} \quad = T \quad {}_{2}R_{§} \ {}^{2}_{§} = {}_{4}R^{§}_{§}\ {}_{§}^{1}\ {}^{1} \quad {}^{2}(\ )L\ {}^{3} \ {}^{2}(\ )L\ {}^{5}$

Details

This Paper (Jump Diffusion)

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Diffusion Volatility

$^{2}(§) :=$ H B1KE $\underset{! \; 0}{}$ R $_{§+}$ $_{§}$ $Lq^{K}(\;)$ ^{2} $F_{§}$

Jump Volatility

$^{2}(§) :=$ H B1KE $\underset{! \; 0}{}$ R $_{§+}$ $_{§}$ $Lq^{`}(\;)$ ^{2} $F_{§}$

| 09e_¶0L K0j _§´I

(Derived Below)

$| K_{§} := T \; \}_{§} \quad {}^{2}_{§} ; {}^{2}_{§} = T$ 2R $_{§}$ $^{2}_{§} = 4$R$^{§}_{§}$ 1 $_{§}$ 1 $^{2}(\;) L$ $^{2}(\;) L$ $^{2}(\;) L$ 3 5 $= i \quad 0;$ Z $_{§}$ $_{§ \; 1}$ $^{2}(\;) L$ $d \quad 0;$ Z $_{§}$ $_{§ \; 1}$ $^{2}(\;) L$

Convolution

Laplace Distribution

Details

How Should We Model Daily Returns?

Recall:

$$r_t | \mathcal{F}_{t-1} \sim p(r_t | \mathcal{F}_{t-1}) = \int_{\sigma_t} f(r_t | \sigma_t) \times g(\sigma_t | \mathcal{F}_{t-1})$$

Sufficient statistic (factor) for the dynamics

How Should We Model Daily Returns?

Recall:

$$`_§ j F_{§ \; 1} \qquad ?(`_§ j F_{§ \; 1}) = \int 7(`_§ j \; t_§) \; / \; : (t_§ j F_{§ \; 1})$$

$^2{}_§$ ← Sufficient statistic (factor) for the dynamics

Two Factors:

$$^2{}_§ = \; {}^2_§ \text{ and } {}^2_§$$

How Should We Model Daily Returns?

Recall:

$$\grave{}_{§}\, j\, F_{§\ 1} \qquad ?(\grave{}_{§}\, j\, F_{§\ 1}) = \int_{^2§} 7(\grave{}_{§}\, j\ t_{§})\ /\ :(t_{§}\, j\, F_{§\ 1})$$

Sufficient statistic (factor) for the dynamics

Two Factors:

$$^{2}{}_{§} = \ {}^{2}_{§} \text{ and } {}^{2}_{§}$$

Model:

$$\}_{§} = \ {}_{§}\, i\,(0;1) + \frac{p\ {}^{§}}{2} d(0;1)$$

$$\begin{matrix} {}^{2}_{§} \\ {}^{2}_{§} \end{matrix} \qquad W(F_{§\ 1})$$

Talk Structure / Contributions (Again)

3 OU Proposed a new measure of jump volatility: $\frac{2}{\S}$.

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Hs 8qqe´ §\0 0 0 §_h9§n} §n _X\]T}0x«0jH´ L9§9 nj §\0 ~ : p U·· q}n®_L_jX h«e§_qe0 j0° §´e_¶0L T9H§ U

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@s ~\n° §\0 $\frac{2}{\S}$ q}0h_«h _ e0 §\9j §\0 $\frac{2}{\S}$ q}0h_«hU

Derive $\grave{}_{§}$ L $0; {}^{2}_{§}$ G $0; {}^{2}_{§}$ from no-arbitrage.

Why does $Lq^{`}(§) = \int_{R} (§; {}^{2})(j \quad ) L^{2} \; 4) \int_{§ \; 1}^{§} Lq^{`}(§) \; d \; 0; \int_{§ \; 1}^{§} {}^{2}(\;) L$?

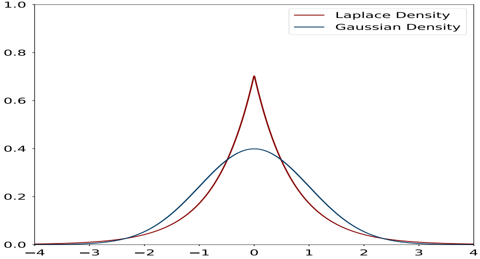
Properties of Laplace Distributions

– PDF: $\frac{1}{2}$ 2 t (T $j^2 j)$

– ~ 2 t (T1); μ $i\,(0; 1)$
4) p ~ μ $d(0; 1)$

– Standardized random sums with a geometrically-distributed number of terms ! geometric-stable, (Klebanov, Maniya, and Melamed 1985; Mittnik and Svetlozar 1993).

– Laplace is a geometric-stable distribution with finite variance.



Standard Variance-Gamma Process

- A variance-gamma process with zero mean and all scale parameters equal to one.

- The variance-gamma process is used to represent processes with frequent jumps.
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- Its increments are i.i.d. Laplace random variables.
  4) ^§ _ 9 dĹ®´ q}nHO u

- "Compound Poisson" arrival-rate, jumps in every interval, finite variance, Gaussian magnitudes
  4) standard variance-gamma

The First Time-Changed Laplace Theorem

Assumptions:

1. $q(\S)$ is a semimartingale. (Delbaen and Schachermayer (1994) showed no-arbitrage is a sufficient condition.)
2. $q^{`}(\S)$ has infinite-activity jumps. ($q^{`}(\S)$ jumps in every interval.)
3. $q^{`}(\S)$ is locally-square integrable. ($\}_{\S}$ has finite variance.)
4. $q^{`}(\S)$ has no predictable jumps.

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$q^{`}(\S)$ §_h 0]H\9jX0L @´ _§ q}0L_H§9@e0 x«9L}9§_H ®9}_9§_nj _ 9 §9jL9}L ®9}_9jH0]X9 h h9 q}nH0 u

Details

Proof Intuition

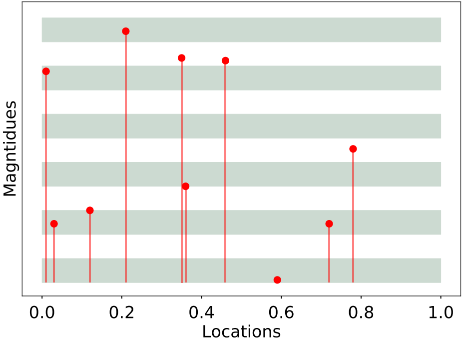
– A jump process has two kind of variation: jump **magnitudes** and jump **locations**.

– Partition the ´-axis.

1. These locations are conditionally independent across strips.
2. They form time-changed Poisson processes, (Time between jumps is distributed 2 t(1).)

– Condition on the jump times in each strip.

1. This is a standardized sum of infinitely-many i.i.d. random variables.
2. The is a time-changed Wiener process. (C.L.T.)



– **Integrating** out the locations and taking limits implies the original process is a **time-changed variance-gamma process**.

What is the Appropriate Time Change?

- We need the **composition** of the previous **time-changes**.

- Changing the magnitude or the intensity affects the variance in the same way.

4) We need the predictable quadratic variation of the original process — $hq^{`}i(\S)$.

- $^{2}(\S)$ is the time-derivative of $q^{`}\ (\S)$.

Details

Continuous-Time Model

Recall:

$$Lq(\S) = (\S)\, L^{-}(\S) + \int_{\mathbb{R}} (\S;\, {}^{2})(j \qquad)(L\S;\, L^{2})$$

Details

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$$Lq(\S) = (\S)\, L^{-}(\S) + \int_{\mathbb{R}} (\S;\ ^{2})(j \qquad)(L\S; L^{2})$$

Allow for drift $(\S)$:

$$Lq(\S) = (\S)\, L\S + (\S)\, L^{-}(\S) + \int_{\mathbb{R}} (\S;\ ^{2})(\qquad)(L\S; L^{2})$$

Details

Continuous-Time Model

Recall:

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Allow for drift $(\S)$**:**

$$Lq(\S) = (\S)\,L\S + (\S)\,L^{-}(\S) + \int_{\mathrm{R}} (\S;\,{}^{2})(\quad)(L\S; L^{2})$$

Simplify jump representation:

$$Lq(\S) = (\S)\,L\S + (\S)\,L^{-}(\S) + \int_{\mathrm{R}} (\S;\,{}^{2})(j \quad)(L\S; L^{2}) \quad : \quad \frac{(\S)}{p\,2}\,Ld(\S)$$

Details

How Should We Model Continuous-Time Returns?

Proposed	**(Conditionally) Exponentially-Affine** (Duffie, Pan, and Singleton 2000; Calvet, Fearnley, Fisher, and Leippold 2015)
$^{2}(\S) := \quad {}^{2}(\S) \quad {}^{2}(\S)$	$^{2}(\S) := \quad {}^{2}(\S) \quad g\,(\S)$

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$Lq(\S) = {}_1 \frac{^2(\S)}{^2(\S)} L\S + (\S) L^{-}{}_1(\S) + \frac{(\S)}{p\overline{2}} Ld(\S)$ $L \frac{^2(\S)}{^2(\S)} = {}_2 \frac{^2(\S)}{^2(\S)} L\S$ $+ {}^{1/2} \frac{^2(\S)}{^2(\S)} L^{-}{}_2(\S)$	$Lq(\S) = {}_1({}^2(\S); g(\S)) L\S + (\S) L^{-}{}_1(\S)$ $+ L\grave{}(\S) \quad \grave{}_{g(\S)} L\S$ $L\ {}^2(\S) = {}_2({}^2(\S)) L\S + H\ (\S) L^{-}{}_2(\S)$ $\grave{}(\S) = \sum_{\S} \grave{}_{g(\S)}\, i_{g(\S)}(\S);\ \grave{}_{g(\S)}(\S) := \mathrm{E}[\grave{}(\S)\, j\, F_{\S}\ ]$ $g(\S)$ follows a Markov-switching process.

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– No latent processes and easy to simulate. – Adapt variance-gamma methods to price options.	– Requires particle filtering. – Affine process price American options in closed-form.

Discrete-Time: ` § = R § § 1 / T(b)

Additional Assumptions:

1. Innovations to prices and volatilities are independent. The drift and volatilities may be correlated.

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\ (}§ j F§ 1) = Z T Z }§ ( ); Z § § 1 2( ); Z § § 1 2( ) LW Z § § 1 ( ); Z § § 1 2( ); Z § § 1 2( ) F§ 1

Estimators

Estimator for $^{2}(\quad)$

Adapted from Jacod (2008) and Aït-Sahalia and Jacod (2009a).

Assumptions:

1. c_j ! 1 .
2. $c_j \, p \overline{_j}$! 0.
3. $q(\S)$ jumps in every interval.
4. $®^{j}_{1}$ tightly bounds the diffusive deviations.
5. Standard technical conditions.

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$$\text{b}^2_j(c_j;\quad;q) := \frac{1}{c_j \quad j} \sum_{h=0}^{c_j 1} \; {}^{j}_{-j} q \; {}^{2} 1 f \; {}^{j}_{-j} q \quad ®^{j}_{1} g \; \overset{P}{!} \; {}^{2}(\quad)$$

What should we use for þ$_1^j$?

Intuition:

- We need a tight bound for the diffusive variation as ! 0.
- The law of the iterated logarithm provides such a bound: H B K b m$_{! 0}$ T $\frac{(\quad)}{(\quad)}$ p $\frac{-(\; + \;) \quad -(§)}{2 \quad \text{H Q H Q 1;/} \quad))}$ = 1:

Algorithm:

1. Use 1.25 times the Bipower estimator for (§).
2. Calculate ®$_1^j$ as implied by the bound above.
3. Then use $b_j^2(c_j; \quad ; q)$
4. Iterate to convergence.

Estimator for $^2(\quad)$

Assumptions:

1. $c_j \to \infty$.
2. $\frac{c_j}{\sqrt{j}} \to 0$.
3. $q(\S)$ jumps in every interval.
4. $\hat{b}_j(\quad) \xrightarrow{P} (\quad)$.
5. Standard technical conditions.
6. $X(\ )$ is a convex weight function.

¦ \0n}0h

$$\hat{b}_j(c_j;\quad;q) := \grave{}\,;\mathrm{KBXM}\left(\frac{1}{c_j\sqrt{j}}\sum_{h=0}^{c_j-1} \quad q^{j}_{-j+h} \quad \mathrm{E} \quad q^{j}_{-j} \quad ; \quad \hat{b}_j^2(\quad)\right) \xrightarrow{P} (\quad)$$

Advantages of $b_j^2(F_j; \quad ; T)$

1. Shows $^2(\S)$ is identified.

2. Enables extending results that depend on instantaneous volatility to the jump case.

   Examples: Mykland and Zhang (2009), Xiu (2010), Barndorff-Nielsen, Hansen, Lunde, and Shephard (2011), Bandi and Renò (2012), and Li, Todorov, and Tauchen (2017)

3. The first consistent estimator for any instantaneous jump variation measure.

Estimators for $\frac{2}{§}$, $\frac{2}{§}$, and $\_ \cdot §$

¦ \0n}0 h

$$\mathrm{b}^2_{§} := \frac{1}{\mathrm{O}\_j \; 2 \; [§ \quad 1; §]} \sum_{§ \quad 1 < §_j \quad §} \mathrm{b}^2(c_j; §_j; q) \; \overset{\mathrm{P}}{!} \int_{§ \quad 1}^{§} {}^2(\;) L$$

¦ \0n}0 h

$$\mathrm{b}^2_{§} := \frac{1}{\mathrm{O}\_j \; 2 \; [§ \quad 1; §]} \sum_{§ \quad 1 < §_j \quad §} \mathrm{b}^2(c_j; §_j; q) \; \overset{\mathrm{P}}{!} \int_{§ \quad 1}^{§} {}^2(\;) L$$

¦ \0n}0 h

N §_h9§0 | $K_§$ @´ qe«XX_jX $\mathrm{b}^2_§$ 9jL $\mathrm{b}^2_§$ _j§n i $0; \int_{§ \quad 1}^{§} {}^2(\;) L$ d $0; \int_{§ \quad 1}^{§} {}^2(\;) L$ u

Simulation Setup

- Set parameters to match the data's discrete-time dynamics.
- Simulate volatilities, $^2(\S)$ and $^2(\S)$, as square-root (Cox-Ingersoll-Ross) processes and the price using the mixture representation.

$$L \;^2(\S) = 9 \;(@ \qquad ^2(\S)) + \qquad (\S)\, L \;^{-}(\S)$$

$$L \;^2(\S) = 9 \;(@ \qquad ^2(\S)) + \qquad (\S)\, L \;^{-}(\S)$$

$$Lq(\S) = \quad (\S)\, L \;^{-}(\S) + \quad (\S)\, Ld(\S)$$

aBKmH iBQM _2bmHib

i i

P#bX T2` JBMX	E[(b$_i$ $_i$)^2]/E[$_i$]				E[(b$_i$ $_i$)^2]/E[$_i$]			
	"La	Ghh	8 JBMmi2	S`	"LaQLTaQb2/ Ghh		8 JBMmi2	S`
2	y.jd	y.9y	:y9y	y.9e	y.dk	:RyR	:3y	y.dk
12	y.j3	y.9y	y.9k	y.Re	y.dy	RyR	y.3j	y.kR
60	y.9y	:y9j	.y98	y.y8	y.e3	:RyR	:3d	y.yd

1 K T B ` B + b

. i

aSu a S 8yy 1h6C Mm `v kyyj a2Ti2K#2` kyRd

hB+F / i 7`QK h Z- b KTH2/ QM+2 T2` b2+QM/X

- jdRj / vbX
- k9i?Qmb M/ Q#b2`p iBQMb T2` / vX
- 3d KBHHBQM iQi H Q#b2`p iBQMbX

_2KQp2/ 2ti` M2Qmb i` /2b M/ mb2/ T`2@ p2` ;BM; iQ }Hi2` Q

- PMHv }M2Hv b KTH2/ / i Bb bm{+B2Mi iQ b2T ` i2 i?2 DmKT M/ /B
- S`2@ p2` ;BM; Bb +QKKQMHv mb2/ iQ ? M/H2 K `F2i@KB+`Qbi`m+

oQH iBHBiB2b _2im`Mb

GQ; oQH iBHBiv .Bbi`B#miBQMb

	HQ;$\frac{2}{i}$	HQ;$\frac{2}{i}$
J2 M	R:yN R	R:ye 9
ai M/ `/ .2pB	R:R j	y:N 3
aF2rM2bb	y:d R	y:8 8
Em`iQbBb	9:R k	j:3 R
CQ``H(Q$\frac{2}{i}$); H(Q$\frac{2}{i}$))	y:N y	

"H +F HBM2b `2 : mbbB M /Bbi`B#miBQMb }i iQ i?2 / i X

. Q 2 b . i = L 0; R i i 1 2 ( ) b / b G 0; R i i 1 2 ( ) b / b r Q ` F r 2 H H \

R X G Q r 2 ` i B H K 2 b m ` 2 / H K Q b i T 2 ` 7 2 + i H v X

k X a F 2 r M 2 b b B b M Q i H ` ; 2 T ` Q # H 2 K B M T ` + i B + 2 X

Z Z S H Q i S ` Q # # B H B i v A M i 2 ; ` H h ` M b 7 Q ` K A U h S A 6 V

h HF ai`m+im`2 f *QMi`B#miBQMb U ; BMV

3 RX S`QTQb2/ M2r K2 bm`2 Q7 ^2_i DXmKT pQH iBHBiv,

Vi? i vB2`iH/ b 0; ^2_i G0; ^2_i

#Vi? i K2 bm`2b DmKTb BM BMp2 bi2 Q`öb BM7Q`K iBQM UM2r bV

3 kX V.2`Bp2/ L 0; $^2_i$ G0; $^2_i$ 7`QK MQ@ `#Bi` ;2X

#V.2`Bp2/ 2biBK i $^2_i$ Q` ^2_i b+ 7MQ/ . $_i$ X

+V TTHB2/ i?2b2 2biBK iQ`b iQ ?B;?@7`2[m2M+v / i QM i?2 a S 8yy

jX Va?Qr ^2_i M/ ^2_i DQBMiHv /2i2`KBM2 `BbF T`2KB X

#Va?Qr i? ^2_i 2T`2KBmK Bb H2b ^2_i b T?2KB mK X

oQH iBHBiv M/ L2rb S`2KB

2 i Bb M2r 7 +iQ` 7Q` i?2 /vM KB+bX

Ab Bi T`B+2/\

S`272`2M+2b U1t KTH2, 1Tbi2BM@wBMV

, `BbF p2`bBQM

, BMi2`i2KTQ` H 2H biB+Biv Q7 bm#biBimiBQM UA1aV

$$I_i = \left[*_i^{1 \quad 1/} + E \left[I_{i+}^{1} \quad F_i \right]^{\frac{1 \quad 1/}{1}} \right]^{\frac{1}{1 \quad 1/}}$$

, `BbF p2`bBQM

, BMi2`i2KTQ` H 2H biB+Biv Q7 bm#biBimiBQM UA1aV

$$l_i = \left[*_i^{1 \quad 1/} + E\left[l_{i+}^{1} \quad F_i \right]^{\frac{1 \quad 1/}{1}} \right]^{\frac{1}{1 \quad 1/}}$$

.2}M@2 $a_i := l_i^{1 \quad 1/}$ X $o_i = \left[*_i^{1 \quad 1/} + E\left[o_{i+}^{\frac{1}{1 \quad 1/}} \quad F_i \right]^{\frac{1 \quad 1/}{1}} \right]$

, `BbF p2`bBQM

, BMi2`i2KTQ` H 2H biB+Biv Q7 bm#biBimiBQM UA1aV

" h i # 1 1 1/ 1 1/ 1 1/ 1
l i = *1 i 1/ + E l1 i+ F i

2 3
1 1/ 1 1 1/ 1
. 2 } M 2 ai := l1 i 1/ X oi = 4 *1 i 1/ + E o1 i+ 1/ F i 5

1 1
. 2 } M 2 (o) := 1 / (1 1/) o 1 1/ : = m(*i) + 1 (E [(oi+) jF i])

AMp2biQ` S`Q#H2K

$$o((i) ; S(i)) = K_{*(i);(i)} t_i^{Z \; i+} m*(b) / b + 2t(T)^{1} (E[(o((i); S(i+))) jF_i])$$

$$*(i) + \sum_B S_B(i) \; (_B i) = \sum_B S_B(i) \; (_B i)$$

bbmKTiBQMb,

RX "Qi?m M/ `2 GBTb+?Bix +QMiBMmQmb rBi? GBTb+?Bix /2`Bp i

kX "Qi?m M/ `2 bi`B+iHv BM+`2 bBM;X

jX *QMbmKTiBQ(*) QMBb M Ai¬ b2KBK `iBM; H2X

9X `2T`2b2Mi iBp2 BMp2biQ` T`B+2b HH bb2ibX

h?2Q`2K

RX G2i i?2 BMp2biQ` 7 +2 i?2 T`Q#H2K 0/2Xb+`B#2/ #Qp2 b

kX bbmK2 T`272`2M+2b `2 bm+? i? i QTiBK H +QMbmKTiBQM Bb

h?2M

$$HQ: \frac{m^0(q(i))}{E[m^0(q(i))]}$$

h i

_BbF S`2KBmK 0/ M CQbp2K(i);BT_B(i) F_i /i

h?2Q`2K

RX G2i i?2 BMp2biQ` 7 +2 i?2 T`Q#H2K 0/2Xb+`B#2/ #Qp2 b

kX bbmK2 T`272`2M+2b `2 bm+? i? i QTiBK H +QMbmKTiBQM Bb

h?2M

$$ HQ: \frac{o^0(q(i))}{E[m^0(q(i))]} E[\,^0(o(q(i)))\,jF_i\,] $$

h i

_BbF S`2KBmK 0/ M C Qbp2 K(i) ;BT:B(i) F i /i

h?2Q`2K

RX G2i i?2 BMp2biQ` 7 +2 i?2 T`Q#H2!K 0/2Xb+`B#2/ #Qp2 b

kX bbmK2 T`272`2M+2b `2 bm+? i? i QTiBK H +QMbmKTiBQM Bb

h?2M

$$HQ: \frac{o^0(q(i))}{E[m^0(q(i))]} E[{}^0(o(q(i)))\, jF_i] \qquad HQ: \frac{{}^0(o(q(i)))}{E[{}^0(o(q(i)))\, jF_i]}$$

h i h i

_BbF S`2KBmK 0/ MC Qp2 K(i) ; BT_B(i) + T_B^C(i) F_i /i CQp K^{IS}(i) ; T_B^C(i) F_i /i

RX BbF T`2KB `2 T`2/B+i #H2 [m /` iB+ p `B iBQMbX UBX2X- 2
`2+m`bBp2 miBHBiv M/ DmKTbXV

kX ^{2}(i) M/ ^{2}(i) DQBMiHv /2i2`KBM2 `BbF@T`2KB X

jXh?2 i?2Q`v `2[mB`2b irQ 7 +iQ`b i? i KQp2 ?B;?@7`2[m2M+v B

CmKT S`QTQ`iBQM $\frac{2_i}{2_i + 2_i}$

oQH iBHBiv *Q``2H iBQMb

	2_i	2_i	$\frac{2_i}{2_i + 2_i}$	`t	1f 6PJ*gi
2_i	Ryy	:yd9	ykN	yRR	yyR
2_i		Ryy	yRy	yRj	yye
$\frac{2_i}{2_i + 2_i}$			Ryy	yRk	yy8
`t				Ryy	yy8

$\frac{2_i}{2_i + 2_i}$

` t = 0 + 1 H Q ; 2 i + 2 i + 2 H Q ; 2 i / (2 i + 2 i) + i

R X / D m b i 7 Q ` ? 2 i 2 ` Q b F 2 / 1 / (2 i + 2 i) b i X + B i v m b B M ;

k X A M b i ` m K 2 M i b ,

G ; ; 2 / 2 i + 2 i M / 2 i / (2 i + 2 i)

S ` 2 / 2 i 2 ` K B M 2 / p ` B # H 2 b ` 2
B M / 2 T 2 M / 2 M i Q 7 B M M Q p i B Q M b X

I b 2 i ? 2 ? 2 i 2 ` Q ; 2 M 2 Q m b m i Q ` 2 ; ` 2 b b B p 2
U > _ V K Q / 2 H H ; b X

_ Q # m b i i Q Q i ? 2 ` + ? Q B + 2 b X

$` t = 0 + 1\ HQ;\ 2_i + 2_i + 2\ HQ;\ \frac{2_i}{2_i + 2_i} + i$

R X/ D m b i 7 Q ` ? 2 i 2 ` Q b F 2 / $\frac{1}{2_i + 2_i}$ b i X + B i v m b i B 2 M ; ? i 2 / @ G 2 b i a [m ` 2 b r B i ? A

k X A M b i ` m K 2 M i b ,

G ; ; 2 / $2_i + 2_i$	M / $\frac{2_i}{2_i + 2_i}$
S ` 2 / 2 i 2 ` K B M 2 / p ` B # H 2 b ` 2 B M / 2 T 2 M / 2 M i Q 7 B M M Q p i B Q M	
l b 2 i ? 2 ? 2 i 2 ` Q ; 2 M 2 Q m b m i Q ` 2 , U > _ V K Q / 2 H H ; b X	
_ Q # m b i i Q Q i ? 2 ` + ? Q B + 2 b X	

_ 2 ; ` 2 b b Q ` b	a T 2 + B } + i B Q M
A M i 2 ` + 2 T i	k N 8 (:ee R)
H Q ; $2_i + 2_i$	y k 9 (:83 3
H Q ; $\frac{2_i}{2_i + 2_i}$	

$$`_t = \;\; _0 + \;\; _1 \, HQ;^2_i + \;\; ^2_i \;\; + \;\; _2 \, HQ; \frac{^2_i}{^2_i + \;\; ^2_i} \;\; + \;\; i$$

RX/Dmbi 7Q` ?2i2`QbF2/$\frac{1}{^2_i + \; ^2_i}$biR +Biv mbBM; ?i2/@G2 bi a[m `2b rBi? A

kXAMbi`mK2Mib,

G ;;2/ $^2_i + \; ^2_i$	M/ $\frac{^2_i}{^2_i + \; ^2_i}$
S`2/2i2`KBM2/ p `B #H2b `2 BM/2T2M/2Mi Q7 BMMQp iBQM	
Ib2 i?2 ?2i2`Q;2M2Qmb miQ`2,	
U> _V KQ/2H H ;bX	
_Q#mbi iQ Qi?2` +?QB+2bX	

_2;`2bbQ`b	aT2+B}+ iBQM
AMi2`+2Ti	kN8 (:ee R)
	k98 (8Rk)
HQ;$^2_i + \; ^2_i$	yk9 (:83 3
HQ; $\frac{^2_i}{^2_i + \; ^2_i}$	8yR (83e)

$$`t = 0 + 1 \, HQ;^{2}_{i} + {}^{2}_{i} + 2 \, HQ;\frac{{}^{2}_{i}}{{}^{2}_{i} + {}^{2}_{i}} + i$$

RX/Dmbi 7Q` ??2i2`QbF2/ $\frac{1}{{}^{2}_{i} + {}^{2}_{i}}$ biX B+Biv mbBM; ?i2/@G2 bi a[m `2b rBi? A

kXAMbi`mK2Mib,

G ;;2/ $^{2}_{i}$ + $^{2}_{i}$ M/ $\frac{{}^{2}_{i}}{{}^{2}_{i} + {}^{2}_{i}}$
S`2/2i2`KBM2/ p `B #H2b `2 BM/2T2M/2Mi Q7 BMMQp iBQM
lb2 i?2 ?2i2`Q;2M2Qmb miQ`2,
U> _V KQ/2H H ;bX
_Q#mbi iQ Qi?2` +?QB+2bX

_2;`2bbQ`b	aT2+B}+ iBQM		
AMi2`+2Ti	kN8 (ee R)	k98 (8Rk)	8y9 (y83)
HQ;$^{2}_{i}$ + $^{2}_{i}$	yk9 (833)		yR9 (ke3)
HQ;$\frac{{}^{2}_{i}}{{}^{2}_{i} + {}^{2}_{i}}$		8yR (83e)	9R8 (9Nj)

h F2 r v

R X* ? M ; 2 b$_i^2$ B+ M$_i^2$ / ` B p 2 H U / Q ; x 2 M b Q 7 +T? 2 ` M ; 2 b i B M ` B b F T ` 2 K B k y R d X B M

k X* ? M ; 2 b $\frac{B_i^2}{+}$ M$_i^2$ / ` B p 2 + ? M ; 2 b B M ` B T b F ` Q t 2 K B i 2 H v i ? 2 b K 2 K ; M B i r

j Xh ? 2 / i ` 2 [ m B ` 2 i r Q 7 + i Q ` b i ? i K Q p 2 ? B ; ? @ 7 ` 2 [m 2 M + v i Q 2 t T

9 XS ` 2 7 2 ` 2 M + 2 b ` 2 M Q i i B K 2 @ b 2 T ` # H 2 X

3 RX S`QTQb2/ M2r K2 bm`2 Q7 2i DXmKT pQH iBHBiv,

Vi? i vB2`iH/ L 0; 2i G0; 2i

#Vi? i K2 bm`2b DmKTb BM BMp2 bi2Q`öb BM7Q`K iBQM UM2r bV

3 kX V.2`Bp2/ L 0; 2i G0; 2i 7`QK MQ@ `#Bi` ;2X

#V.2`Bp2/ 2biBK iQ`b 7MQ/ .i X

+V TTHB2/ i?2b2 2biBK iQ`b iQ ?B;?@7`2[m2M+v / i QM i?2 a S 8yy

3 jX Va?Qr2/ M/ DQBMiHv /2i2`KBM2 `BbF T`2KB X

#Va?Qr T`2KBmK Bb H2b i?2 KB ?m2KX

p2Mm2b 7Q` 6mim`2 qQ`F

RX J2 bm`2 i BH `BbF BM `2 H@iBK2- a M;`2v UkyR3#- qSVX

Vh?Bb T T2` `2/m+2b 7Q`2+ biBM; i?2 /2MbBiv iQ 7Q`2+ biBM; i?2

#VA +QMbBbi2MiHv 2biBK i2/ i?2 pQH iBHBiB2b M/ b?Qr2/ i?2v `2 : mbbB MX

4) q2 + Mi` +F M/ 7Q`2+ bi bi M/ `/ `BbF K2 bm`2b bm+? b o Hm2@ `2 bi iBbiB+b Q7 i?2 / BH? (`jF$_i$ m$_{i1}$)) X /2MbBiv

kX .2p2HQT KmHiBp `B i2 p2`bBQM Q7 i?Bb T T2`X *QMD2+im`2 pQH iBHBiv /2T2M/2M+2X

jX .2p2HQT i?2 TT`QT`B i2 MQBb2@`Q#mbi BM72`2M+2 i?2Q`vX

9X q?v /Q2b $\frac{2_i^2}{2_i^2 + 2_i^2}$ +QKK M/ bi iBbiB+ HHv M/ 2+QMQKB+ HHv M2; iBp2 bB;MB7B+ Mi rQmH/ i?BMF i BH@`BbF K2 bm`2 rQmH/ +QKK M/ TQbBiBp2

CmKT hBK2b `2 L2rb hBK2b

h?2Q`2K

*QMbB/2` biQT XBG M2(i)S#K2 T`B+2 T`Q+2bb b iBb7vBM; MQ@ `#Bi }Hi` iBQFM +QMi BMb HH Q7 i?2 BM7Q`K iBQM BM i?2 `2T`2b2Mi `2H2p Mi 7Q` bb2i T`B+BM;- M/ (Q M) HD m K T b- i?2 Bb i?2 bbQ+B T`2/B+i #H2 }Hi` iBQMX

" Z i+ #

2 i = HBKE 1 / T(b) 2 F i

! 0 i

2 2 33

= HBKE 1 4 E 4 X (DmKT K ;M) 2 (Bi) m F/ i; 2bBb DmKT 55

! 0

b2(i; i+]

= HBKE 1 (DmKT K ;M) 2 BE[mO/Q7 DmK(T; b+B] M

! 0

h?2Q`2K UhBK2@*? M;BM; CmKT J `iBM; H2bV

G2i(T) #2 M BM}MBi2Hv@ +iBp2- Tm`2Hv /Bb+QMiBMmQmb- b[m
mMT`2/B+i #H2 DmKTb i? i + M (#M2 `)2 r?2b2 (MB2b/ bT`>2/B+i #H2 T`Q
SQBbbQM ` M/QK K2BbbmT`22/BM+i #H2 +QKT2Mb iQ` rBi? G2#2b;m2
h?2M(i)TiBK2@+? M;2/ #v Bib T`2/B+i #H2 [m /` iB+ p `B iBQM Bb
Qi?2` rQ`(/i)b= TGhTi(i) X

hBK2@*? M;2/ G TH +2 *Q`QHH `B2b

*Q`QHH `v

G2i (i)Ti #2 Tm`2Hv @/Bb+QMiBMmQmb Ai¬ b2KBK `iBM; H2 i? i Bb BM}MBi2@ +iBpBiv (i)D=m$\frac{1}{2}$K TR_i b(X) hG2-M ?T2`2 G Bb bi M/ `/ p `B M+2@ M/(i) Bb T`2/B+i #H2 7mM+iBQMX

*Q`QHH `v

G2i (i)T #2 Tm`2Hv /Bb+QMiBMmQmb- HQ+ HHv@b[m `2 BMi2;` #H > ( ) r?2`2 (i)>Bb T`2/B+i #H2 TS`QB+2bb0Q+M ` M/QK K2BbmT`22/BM+/i +QKT2Mb iQ` rBi? G2#2b;m2 #Xb6mG`2?p2v`K2bbmK`2 i? i Bi ? b MQ T` h?2M(i)TiBK2@+? M;2/ #v Bib T`2/B+i #H2 [m /` iB+0pT`Q+i2bbQM/ B?b2 bi M/ `/ p `B M+2@; KK T`Q+2bb r?2`2 i?2 KBtBM; r2B;?ib `2 /2 T`Q+2bbX

1biBK iB^2(M);

h?2Q`2K

G2(i)T#2 HQ+ HHv@b[m `2 BMi2;` #H2 BM}MBi2@ +iBpBiv Ai¬ b
HQ+ HHv@b[m `2 BMi2;` #H2 !1¨/H ¨F; /2 M b0BiBM2 b X h<G1 2 i#F2 /2i2`KB
iBK2X .2}MB2 BF 1X G^2 i +QMp2`;2 BM T`Q #X#G 2(i) >v0 iTQ` HH i- i?2
? p2 i?2 7QHHQrBM;X

$$\hat{\;} := \; `;KB M \frac{1}{F_M} \sum_{K=0}^{F_M - 1} \frac{M}{B_M + K} \quad s \quad Ej L(0;1)j\hat{\;}_h \quad \frac{p}{2} \; 2`7+t \frac{\hat{\;}_h}{\;} \quad !^P \; h$$

GQ+ H #bQHmi2 o `B iBQM

h?2Q`2K

G2i (T) #2 HQ+ HHv@b[m `2 BMi2;` #H2 BM}MBi2@ +iB pBiv Ai¬ b2
HQ+ HHv@b[m `2 BMi2;` #H2 !1¨/H¨M //2M 0B 2 bX <Ch2< 1F #2
/2i2`KBMBbiB+ iB= K 2BXF 2} XM 2 B

$$\frac{1}{\sqrt{F_M}} \sum_{K=0}^{F_M - 1} \ldots \; E\,|\,L(0;1)\,| \; h \; + \; 2`7 + \sqrt{\frac{h}{2}} \; h$$

aBKmH iBQM _2bmHib rBi? JB+`Qbi`m+im`2

P#bX f JB	Mmi2	$E[(b_i\ \ )^2]/E[\ \ _i]$				$E[(b_i\ \ )^2]/E[\ \ _i]$			
		"BTQr2`		Ghh	8 JBMmi2	"BTQr2` / S`QT`Q`b2		Ghh	8 JBMmi2
2		y.d9	y.9R	y.9k	R.yy	R.yR	:.yRR	:.y3j	y.e8
12		y.3k	y.9e	y.9e	y.je	R.yR	R.yR	y.3k	y.9R
60		R.RR	:.eyN	:.eyN	y.je	R.yR	:.yRR	:.y39	y.kR

i i

P#bX T2` J	BMX	$E[(b_i\ i)^2]/E[\ i]$				$E[(b_i\ i)^2]/E[\ i]$			
		"La	Ghh	8 JBMmi2	S`	"QLTaQbG/	hh	8 JBMmi2	S`
2		y333	yRk	yky	y333	RyR	:yRR	:yd3	yj9
12		yN8	yRj	ykR	y8R	RyR	RyR	ydN	yjk
60		RRd	:jyk	y9R	yyN	RyR	:yRR	:3yy	yjN

$$`t = {}_0 + {}_1 \, HQ \, ;^{2}_{i} + {}^{2}_{i} + {}_2 \, HQ \, ; \frac{{}^{2}_{i}}{{}^{2}_{i} + {}^{2}_{i}} + {}_3 \, HQ \, ;^{2}_{i} + {}_4 \, HQ \, ;^{2}_{i} + {}_i$$

L2rb S`2KB 1biBK i2b

_2;`2bbQ`b	aT2+B}+ iBQMb					
AMi2`+2Ti	kN8 (:ee R)	k98 (8Rk)	8y9 (y83)	:jkd (:kd8)	:kN8 (:yed)	:jkk (:R98)
$HQ \, ;^{2}_{i} + {}^{2}_{i}$	y:k9 (:ee R		y:R9 (:ke3			
$HQ \, ; \frac{{}^{2}_{i}}{{}^{2}_{i} + {}^{2}_{i}}$		8yR (83e)	9R8 (9Nj)			
$HQ \, (;^{2}_{i})$				y:k8 (:e8j)		R:3e (:8R3)
$HQ \, ;^{2}_{i}$					y:kj (:89y)	R:d9 (98j)

$$`t = {}_0 + {}_1 HQ;_i^2 + {}_i^2 + {}_2 HQ;\frac{{}_i^2}{{}_i^2 + {}_i^2} + {}_i$$

PGa

_2;`2bbQ`b	aT2+B}+ iBQMb			
AMi2`+2Ti	988 (83R)	:Rk (:9e)	j:Rd (:N9)	Rkd (y89)
$HQ;_i^2 + {}_i^2$	y9e (883)		yjN (9Rk)	yRN (y38)
$HQ;\frac{{}_i^2}{{}_i^2 + {}_i^2}$		Re8 (83R)	:Rj (y9e)	:jNR (yRd)
$HQ;_i^2 + {}_i^2 \quad HQ;\frac{{}_i^2}{{}_i^2 + {}_i^2}$				ykN (y3y)
R^2	ked W	:eRR W	:j8 W	:9k W

B M/ " Mbk yHRV3 lb2 6PJ* / i2b iQ T`Q $\frac{2_i}{2_i + 2_i}$ tv X7Q`

1f 6PJ*g_i $\frac{2_i}{2_i + 2_i}$

A1bf 6PJg*i Bb ;QQ/ T`Q $\frac{2_i}{2_i + 2_i}$ 7Q`

_2;`2bb p = M_0/+ $_1$1f 6PJg*i + i		
_2;`2bb M/	HQ; $\frac{2_i}{2_i + 2_i}$	HQ; 2_i + 2_i
AMi2`+2Ti	R:ye8 (RR:3de)	R:yye (R:y89)
1f 6PJg*i	y:8y (:dNy)	y:98 (:de3)
R^2	y:d3W	y:83W
O1f 6PJg*i = 1	RR8	

oQH iBHBiv amKK `v ai iBbiB+b

	2_i	2_i	$\frac{2_i}{2_i + 2_i}$	HQ(2_i)	HQ(2_i)	HQ;(2_i + 2_i)	HQ;($\frac{2_i}{2_i + 2_i}$)
J2 M	9:9d Ry^8	j:e3 Ry^8	y:8e	R:yNR	R:ye9	R:jR8	k:Rd
ai/X .2pX	R:8k Ry^9	N:Rk Ry^8	y:Rk	R:Rj	y:N3	R:yj	y:kk
aF2rX	R:8e8	R:R3R	y:R3	:ydR	y:88	y:dk	y:N8
Em`	jd:e88	k8:ykj	k:Nk	9:Rk	j:3R	9:Ry	9:33